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Comparative Dynamics (Sensitivity Analysis) in Optimal Control Theory*

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This paper considers an optimal control problem with a parameter and develops a systematic method for comparative dynamics. A sufficient condition for the optimum solution to be differentiable with respect to the parameter is provided. Formulas for computing the derivative are given in the form of initial-value problems of linear differential equations. The possibility of discontinuous optimal controls is fully taken care of. An example of the comparative dynamics is given in terms of a model of optimal capital accumulation.

I. INTRODUCTION

Optimal control theory was developed by Pontryagin and his associates [9] as a renovation of the classical theory of calculus of variations. It provides a convenient method for analyzing a wide class of economic problems such as planning the optimal capital accumulation for an economy and investigating the process of investment by a firm or by an individual. Although many applications of Pontryagin's theory to economic problems have been published, only a few of them have paid attention to the problem of comparative dynamics (sensitivity analysis), i.e., that of analyzing the effect of a parameter on the optimum solution.[1]

* An earlier version of a part of this paper was included in the author's doctoral dissertation submitted to Stanford University, 1968. I owe much to Professor Kenneth J. Arrow, especially for the results obtained in Section IV.

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[1] See, e.g., Koopmans [6, pp. 119–122], for a graphical approach to comparative dynamics in a model of optimal economic growth. Cass [3, p. 844] also dealt with a comparative dynamics in optimal growth. Jorgenson [5, pp. 147–151] suggested a method for comparative dynamics in investment theory. Oniki [7 and 8] made extensive use of the method presented in this paper for solving problems of optimal growth and human investment, respectively. Outside the economics literature, Barriere [2] set forth investigating the subject. His viewpoint, however, is narrower than that of this paper:

In planning problems, comparative dynamics may be useful for investigating the dependence of a plan on exogenous factors. In dynamic behavioral models, it could serve as a tool for deriving the intertemporal demand or supply functions of commodities.

A systematic method is developed for dealing with comparative dynamics in optimal control problems. To facilitate the basic idea, let us consider a simple problem of maximizing the function $f(x, \Theta)$ in x for a given value of Θ, which is a parameter. What we usually do is to obtain the first-order condition

$$f_x(x, \Theta) = 0, \tag{1}$$

and to solve this for x. Assume that the solution x thus obtained is a unique optimum. The dependence of this optimum on the parameter Θ may be studied by calculating the derivative of x with respect to Θ:

$$f_{xx} \cdot x'(\Theta) + f_{x\Theta} = 0 \qquad \text{or} \qquad x'(\Theta) = -(f_{xx})^{-1} \cdot f_{x\Theta}\,. \tag{2}$$

This method can be extended to constrained maximum problems. In the theory of household behavior, for example, the solution is known in such terms as Slutzky equations, Hicksians, etc., where the role of Θ is played by income or the commodity prices.

It is noted that Pontryagin's optimum condition for a control problem (composed of the maximum principle, auxiliary differential equations, and transversality conditions) is essentially the first-order condition; in this sense, it is an extension of (1).[2] What we intend to do in this paper under the name of comparative dynamics is to extend (2) to optimal control problems. Under certain assumptions, we shall provide a set of formulas by means of which the derivatives of the optimum solution of a control problem with respect to a parameter may be calculated.

It is well known that under certain conditions a solution of a system of differential equations is differentiable with respect to a parameter appearing in the system (the theorem of variational differential equations).[3] Since Pontryagin's condition contains differential equations, it is suggested that one might make use of this theorem for comparative dynamics. However,

first, he analyzed the effects on the objective function only, while we cover those on the optimum control and the state variables as well; second, he did not state any condition sufficient for the objective function to be differentiable with respect to a parameter, while we do.

[2] In fact, the maximum principle contains more than the first-order condition in the calculus of variations does. In this paper, however, we do not exploit the implications of the maximum principle beyond those expressed in terms of first-order derivatives.

[3] See Lemma 1 of Section III.

an immediate application is difficult, since the theorem presupposes that the differential equations are continuous in state variables, while those appearing in Pontryagin's condition frequently exhibit discontinuities in state variables (e.g., bang-bang controls).[4] To resolve this difficulty, we extend the theorem of variational equations to the discontinuous case (Section III). Once this is attained, we can readily derive formulas for comparative dynamics from Pontryagin's condition (Section IV).

The method of comparative dynamics to be presented in this paper is sufficiently general to allow a parameter to appear almost anywhere in the original control problem; e.g., the objective function may contain it, the initial state may be a function of it, or the constraint on the control may be affected by it (or any combination of these). The paper deals with the effect of a parameter on the optimal control, on the state variables, and on the objective function. Both cases of finite and infinite horizons are considered.

In the following section (Section II) we formulate the problem. Section III is devoted to extending the theorem of variational differential equations to the case in which the differential equations are discontinuous in the state variables. The main results will be presented in Section IV. In that section, a set of conditions sufficient for the optimum solution to be differentiable with respect to a given parameter will be stated, together with formulas for computing the derivative. The last section (Section V) is devoted to an example in which the method of comparative dynamics is applied to a model of optimal capital accumulation.

II. The Problem

We shall be concerned with an optimal control problem which contains a parameter. The problem, for a given value of the parameter, is one including fixed time-horizons, variable end points, autonomous differential equations, and fixed constraints on controls.[5,6] The objective function,

$$v = \int_{t_0}^{t_1} f^0(x(t), u(t), \Theta)\, dt, \tag{3}$$

[4] Cass [3, footnote 5] used this theorem for comparative dynamics. In his case, however, the optimal control is continuous in state variables, and the difficulty stated in the text does not arise.

[5] The assumption that the differential equations are autonomous is not restrictive, since a nonautonomous system can always be converted into an autonomous system by introducing an additional state variable, $x_{n+1} \equiv t$, say.

[6] For simplicity, we assume that for each Θ the constraints on controls are fixed so that g in (5) is independent of x. The main results of the paper will continue to hold if $g(u, \Theta)$ is replaced by $g(x, u, \Theta)$ (with appropriate modifications, some of which will be mentioned later in footnotes).

is maximized in $x(t)$ and $u(t)$, subject to

$$\dot{x}(t) = f(x(t), u(t), \Theta), \tag{4}$$

$$g(u(t), \Theta) \geqslant 0 \qquad (t_0 \leqslant t \leqslant t_1), \tag{5}$$

$$\zeta^i(x(t_i), \Theta) = 0 \qquad (i = 0, 1), \tag{6}$$

where $(t_0, t_1)(-\infty \leqslant t_0 < t_1 \leqslant +\infty)$ is a fixed interval of time, x is an n-vector (*the state variables*), u is an m-vector (*the controls*), Θ is a number (*the parameter*), and the functions $f^0 : R^{n+m+1} \to R^1$, $f : R^{n+m+1} \to R^n$, $\zeta^i : R^{n+1} \to R^{r_i}(0 \leqslant r_i \leqslant n)$, $g : R^{m+1} \to R^k(k \geqslant 0)$ are all assumed continuously differentiable, R^j being a j-dimensional space. Let us put

$$\begin{aligned} \dot{v}(t) &= f^0(x(t), u(t), \Theta) \qquad (t_0 \leqslant t \leqslant t_1), \\ v(0) &= 0, \end{aligned} \tag{7}$$

so that

$$v = v(t_1). \tag{3a}$$

If the functions $u(t, \Theta)$, $x(t, \Theta)$, and $v(t, \Theta)$ maximize (3a) subject to (4)–(7) for a given parameter Θ, they are called an *optimum*. In addition, we call such $u(t, \Theta)$ an *optimal control.*

The appearance of Θ in the optimum functions reflects the fact that, in general, an optimum depends on it. Comparative dynamics deals with how a change in Θ affects an optimum. First of all, it can easily be established that for each t the set of optimum solutions $u(t, \Theta)$ and $x(t, \Theta)$ is upper semicontinuous in Θ (with respect to the set inclusion relation), since everything is continuous in Θ. Furthermore, the objective function

$$v = v(t_1) = v(t_1, \Theta), \tag{3b}$$

is continuous in θ. In the present paper, we focus our attention on the differentiability of an optimum solution with respect to θ, assuming that an optimum is unique.

It is seen that without losing generality the differentiability of an optimum solution may be examined with an additional assumption: $\Theta = 0$. For simplicity, when Θ is set equal to zero, we may suppress the number 0 for Θ in the argument of a function. Thus, $f(x, u) = f(x, u, 0)$, $x(t) = x(t, 0)$, etc.

Next, we state Pontryagin's optimum condition for the control problem.[7,8] First of all, we define the Hamiltonian,

$$H(p, x, u, \Theta) = f^0(x, u, \Theta) + p \cdot f(x, u, \Theta), \tag{8}$$

and its maximum in u subject to (5),

$$M(p, x, \Theta) = \max_{g(u,\Theta)\geqslant 0} H(p, x, u, \Theta), \tag{9}$$

where p is an n-vector (*the auxiliary variables*) and the center dot in (8) denotes the inner product. If $u(t)$ and $x(t)$ is an optimum for a problem in which $\Theta = 0$, then there exists a nontrivial function $p(t)$ on (t_0, t_1) such that

the maximum principle,

$$H(p(t), x(t), u(t)) = M(p(t), x(t)), \tag{10}$$

the auxiliary differential equations are

$$\dot{p}(t) = -H_x(p(t), x(t), u(t)) \qquad (t_0 \leqslant t \leqslant t_1), \tag{11}$$

and the transversality conditions,

$$p(t_i) \in L^i = L^i(x(t_i)) \qquad (i = 0, 1), \tag{12}$$

are satisfied, provided that

$$\operatorname{rank}(\zeta_x^i(x(t_i)) = r_i \qquad (i = 0, 1), \tag{13}$$

where H_x and ζ_x^i are, respectively, the partial derivatives of H and ζ^i with respect to x; H_x will be treated as an n-vector and ζ_x^i as an (r_i, n)-matrix. Further, L^i is the subspace of R^n spanned by the row vectors of $\zeta_x^i(x(t_i))$.

The optimum solution characterized by the above condition ranges over a very wide variety. In order to isolate a class of optimum solutions

[7] See Pontryagin [9, pp. 66–69, 189–191]. If the time-horizon is infinite so that $t_1 = +\infty$, then we assume that $\lim_{t_1\to+\infty} x(t_1)$ exists. (Similarly, for the case of $t_0 = -\infty$.) Furthermore, an optimum solution of the infinite-horizon problem might not satisfy Pontryagin's condition (See Arrow and Kurz [1, p. 46]). In the following, we exclude such cases from our consideration.

[8] If the function g depends not only on (u, Θ) but also on x, then (9) is modified accordingly, and (10) is replaced by $\dot{p}(t) = -[H_x(p(t), x(t), u(t)) + \lambda(t) g_x(x(t), u(t))]$, where $\lambda(t)$ is the Lagrangian multipliers associated with g in (9). With these modifications, the following discussions will continue to hold.

which can be studied by means of comparative dynamics, we define a regular optimum in the following way:

DEFINITION. (a) A pair (p, x) is *regular at* Θ, if the control u satisfying the maximum principle,

$$H(p, x, u, \Theta) = M(p, x, \Theta), \tag{16}$$

is unique and is a continuously differentiable function,

$$u = u(p, x, \Theta), \tag{15}$$

in a neighborhood of (p, x, Θ).

(b) An optimum control $u(t, \Theta)$ is *regular at* Θ *and* t, if $(p(t, \Theta), x(t, \Theta))$ is regular at Θ and $u(t, \Theta) = u(p(t, \Theta), x(t, \Theta), \Theta)$.

(c) An optimum control $u(t, \Theta)$ is *regular on* (t_0, t_1) *at* Θ, if the optimum control is regular at Θ and at t_0, at t_1, and at each t of (t_0, t_1) but a *finite* number of points, say s_j $(j = 1,..., q;\ q \geq 0)$ (s_j's being called *switching time-points*), where $t_0 < s_1 < s_2 < \cdots < s_q < t_1$.

It is easily seen that not all optimum solutions are regular. In economic applications, however, optimum solutions usually turn out to be regular. In the sequel, we shall be concerned only with regular optimum solutions.

The concept of regularity defined above has a close relation to (in fact, is originated from) the method of phase-diagrams, which is widely used for solving control problems. The following is a typical regular optimum for a problem with $\Theta = 0$, which will be described in terms of phase-diagrams: The entire interval (t_0, t_1) is divided into $(q + 1)$ subintervals by the switching time-points, $s_1, ..., s_q$. We consider an optimal control $u(t)$ which is regular on (t_0, t_1) at $\Theta = 0$. It is smooth in the interior of any subinterval, but it may be discontinuous at a switching time-point. On the other hand, the (p, x)-space is divided into regions of regular points, each region being an open set (from definition (a) of regularity). The path of the optimum solution $(p(t), x(t))$ starts at an interior point of a region at $t = t_0$, crosses its boundary at $t = s_1$, stays in the interior of another region for $s_1 < t < s_2$, crosses its boundary at $t = s_2$, and so on. It terminates at an interior point of some region at $t = t_1$. The path is smooth in the interior of any region. At a boundary point it is continuous but may not be smooth. If we denote the boundary that the path crosses at $t = s_j$ by the equation $h^j(p, x) = 0$, then

$$h^j(p(s_j), x(s_j)) = 0 \qquad (j = 1,..., q). \tag{16}$$

We call $(p(s_j), x(s_j))$ *a switching point*. Figures 1a and 1b illustrate a regular optimum solution for a case of $n = m = 1$ and $q = 2$.

An optimum control "switches" at a time-point, say $t = s$, for various reasons. Switching might occur because the set of effective constraints on controls is changed from one to another at $t = s$. Also, it might occur because the control satisfying (14) jumps from a local maximum to another local maximum at $t = s$. We study the properties of the optimum control at a switching point in more detail in Section IV.

We are now able to state the problem. First, to simplify the notation, let us introduce

$$\begin{aligned} z &= (p, x), \\ F(z, \Theta) &= (-H_x(z, u(z, \Theta), \Theta) \qquad f(x, u(z, \Theta), \Theta)), \end{aligned} \tag{17}$$

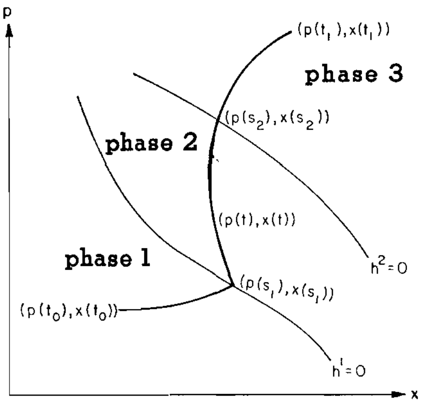

FIG. 1a. Example of a regular optimum solution.

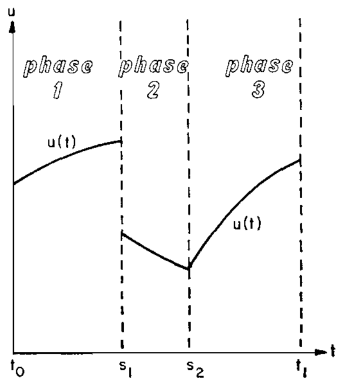

FIG. 1b. Example of a regular optimum control.

where z is a $(2n)$-vector and $F : R^{2n+1} \to R^{2n}$ is a continuously differentiable function for regular $z = (p, x)$. It may be discontinuous at a point z satisfying $h^j(z) = 0$. Then, the constraints (4)–(6), the optimum condition (10)–(12), and the switchings (16) for a problem in which $\Theta = 0$ can be reduced to *the original and the auxiliary differential equations together with the maximum principle*,

$$\dot{z}(t) = F(z(t)), \tag{18}$$

for regular t in (t_0, t_1) (i.e., for all t but $s_1, ..., s_q$);
the switchings,

$$h^j(z(s_j)) = 0 \qquad (j = 1, ..., q); \tag{19}$$

and *the end and the transversality conditions*,

$$\Psi^i(z(t_i)) = 0 \qquad (i = 0, 1), \tag{20}$$

where $\Psi^i : R^{2n+1} \to R^n$ is a function of (z, Θ) and (20) is equivalent to (5) *and* (12). To see this, it suffices to observe from (13) and the definition of L_i that (12) is equivalent to $(n - \boldsymbol{r}_i)$ linear constraints on $z(t_i)$.

Our task will then be to investigate how the function, $z(t) = z(t, 0)$, is shifted when the parameter Θ is changed near $\Theta = 0$. In Section IV, we obtain a set of conditions sufficient for $z(t, \Theta)$ to be differentiable with respect to Θ at $\Theta = 0$ and formulas to compute the derivative.

It is noted that the condition (18)–(20) is but a necessary condition for optimum. Hence it is certainly possible that $z(t) = z(t, 0)$ satisfying (18)–(20) is an optimum but $z(t, \Theta)$ satisfying equations like (18)–(20) for a small $\Theta \neq 0$ is not an optimum. If this is the case, the derivative of $z(t, \Theta)$ with respect to Θ at $\Theta = 0$, though it exists, does not describe the shift of the optimum solution. Such a case may arise, even if the optimum for $\Theta = 0$ is regular. (If the optimum is not unique at $\Theta = 0$, then usually this will be the case.) It seems that there is no systematic method for dealing with this kind of complexity. In applications, however, it is frequently the case that the function $z(t, \Theta)$ satisfying (18)–(20) is the *unique* optimum for *all* Θ near $\Theta = 0$. The method of comparative dynamics to be presented below can be used in such cases.

III. Variational Differential Equations

In this section, we state two lemmas on variational differential equations. The first is well known in the theory of ordinary differential equations. The second is an extension of the first to the case in which the differential equations are discontinuous in the state variables. The notation in this section is independent of that in the previous and the following sections.

Consider the following system of differential equations and an initial condition, both containing a parameter Θ:

$$\dot{x} = F(x, \Theta), \tag{21}$$

$$x(\tau(\Theta)) = \xi(\Theta), \tag{22}$$

where x and ξ are vectors, F a vector-valued function; τ denotes the initial time, and ξ the initial point of the state variable x. The functions F, τ, ξ are all continuously differentiable. Further, F_x, F_Θ, τ_Θ, etc., are partial derivatives, and $F(x, 0) = F(x)$, $\xi(0) = \xi$, etc. It is assumed that a fixed interval of time $T = (t_0, t_1)$ is given, and that

$$t_0 < \tau(\Theta) < t_1 . \tag{23}$$

LEMMA 1. (Peano). *Suppose that a solution $x(t)$ of (21) and (22) exists for $\Theta = 0$ on the entire interval T:*

$$\dot{x}(t) = F(x(t)), \qquad (t \in T), \tag{21a}$$

$$x(\tau) = \xi. \tag{22a}$$

Then, there exists a positive number, say $\hat{\Theta}$, for which the following is *true*: (i) *For each $\Theta(|\,\Theta\,| < \hat{\Theta})$, a unique solution $x(t, \Theta)$ of (21) and (22) exists on T;*

(ii) *For each $\Theta(|\,\Theta\,| < \hat{\Theta})$ and each $t(t \in T)$, the following expressions exists and are continuous in (t, Θ):*

$$\begin{aligned} \dot{x}(t, \Theta) &= \partial x(t, \Theta)/\partial t, \\ x_\Theta(t, \Theta) &= \partial x(t, \Theta)/\partial \Theta, \\ \dot{x}_\Theta(t, \Theta) &= \partial^2 x(t, \Theta)/\partial t\, \partial \theta = \partial^2 x(t, \Theta)/\partial \Theta\, \partial t; \end{aligned} \tag{24}$$

(iii) *The derivative $x_\Theta(t) = x_\Theta(t, 0)$ satisfies the system of variational equations,*

$$\dot{x}_\Theta(t) = F_x(x(t))\, x_\Theta(t) + F_\Theta(x(t)), \text{ on } T, \tag{25}$$

and the initial condition,

$$x_\Theta(\tau) = \xi_\Theta - \dot{x}(\tau)\, \tau_\Theta .^9 \tag{26}$$

[9] For a proof, see Pontryagin [10, pp. 170–177, 194, 198] or Hartman [4, pp. 93–94, 95–100].

Next, let $h(x, \Theta)$ be a scalar-valued function. We deal with the following system of differential equations with discontinuities in the state variables:

$$\begin{aligned} \dot{x} &= F(x, \Theta), \qquad \text{if} \quad h(x, \Theta) \geqslant 0, \\ &= G(x, \Theta), \qquad \text{if} \quad h(x, \Theta) < 0, \end{aligned} \tag{27}$$

$$x(\tau(\Theta)) = \xi(\Theta), \tag{28}$$

where F and G are vector-valued functions. We assume that the functions F, G, h, τ, and ξ are all continuously differentiable with respect to (x, Θ) or Θ. We assume, for definiteness, that

$$h(\xi(0), 0) = h(\xi) > 0. \tag{29}$$

LEMMA 2. *Suppose that a solution* $x(t)(t \in T)$ *of* (27) *and* (28) *exists for* $\Theta = 0$. *The function* $x(t)$ *satisfies*

$$x(t) \text{ is continuous on } T, \tag{30}$$

and there exists a switching time $s(t_0 < s < t_1)$ *such that*

$$\dot{x}(t) = F(x(t)) \quad \text{and} \quad h(x(t)) > 0 \quad \text{for} \quad t_0 \leqslant t < s; \tag{31}$$

$$h(x(s)) = 0; \tag{32}$$

$$\dot{x}(t) = G(x(t)) \quad \text{and} \quad h(x(t)) < 0 \quad \text{for} \quad s < t \leqslant t_1; \tag{33}$$

$$x(\tau) = \xi. \tag{34}$$

Suppose further that

$$h_x(x(s)) \cdot F(x(s)) \neq 0 \quad \text{and} \quad h_x(x(s)) \cdot G(x(s)) \neq 0, \tag{35}$$

where the dot denotes the inner product.

Then, there exists a positive number, say $\hat{\Theta}$, *for which the following is true*:

(i) *For each* $\Theta(|\,\Theta\,| < \hat{\Theta})$, *there uniquely exists a switching time* $s(\Theta)$ *and a solution* $x(t, \Theta)$ *of* (27) *and* (28) *on* T. *The functions* $s(\Theta)$, $x(t, \Theta)$ *satisfy*:

$$\dot{x}(t, \Theta) = F(x(t, \Theta), \Theta) \text{ and } h(x(t, \Theta), \Theta) > \text{ for } t_0 \leqslant t < s(\Theta); \tag{36}$$

$$h(x(s(\Theta), \Theta), \Theta) = 0; \tag{37}$$

$$\dot{x}(t, \Theta) = G(x(t, \Theta), \Theta) \text{ and } h(x(t, \Theta), \Theta) < 0 \text{ for } s(\Theta) < t \leqslant t_1; \tag{38}$$

$$x(\tau(\Theta), \Theta) = \xi(\Theta). \tag{39}$$

(ii) *For each* $\Theta(|\Theta| < \hat{\Theta})$ *and each* $t(t \in T, t \neq s(\Theta))$, *the following derivatives exist and are continuous in* Θ *or* (t, Θ):

$$\begin{aligned} s_\Theta(\Theta) &\equiv ds(\Theta)/d\Theta, \\ \dot{x}(t, \Theta) &\equiv \partial x(t, \Theta)/\partial t, \\ x_\Theta(t, \Theta) &\equiv \partial x(t, \Theta)/\partial \Theta, \\ \dot{x}_\Theta(t, \Theta) &\equiv \partial^2 x(t, \Theta)/\partial t\, \partial \Theta = \partial^2 x(t, \Theta)/\partial \Theta\, \partial t. \end{aligned} \tag{40}$$

(iii) *The derivatives* $x_\Theta(t) \equiv x_\Theta(t, 0)(t \neq s)$ *and* $s_\Theta \equiv s_\Theta(0)$ *satisfy*:

$$\dot{x}_\Theta(t) = F_x(x(t))\, x_\Theta(t) + F_\Theta(x(t)) \qquad \text{for} \quad t_0 \leqslant t < s; \tag{41}$$

$$x_\Theta(\tau) = \xi_\Theta - x(\tau)\, \tau_\Theta\,; \tag{42}$$

$$\dot{x}_\Theta(t) = G_x(x(t))\, x_\Theta(t) + G_\Theta(x(t)) \qquad \text{for} \quad s < t \leqslant t_1\,; \tag{43}$$

$$x_\Theta(s + 0) = x_\Theta(s - 0) - [\dot{x}(s + 0) - \dot{x}(s - 0)]\, s_\Theta\,; \tag{44}$$

$$\begin{aligned} s_\Theta &= -[h_x(x(s)) \cdot x_\Theta(s - 0) + h_\Theta(x(s))/h_x(x(s)) \cdot \dot{x}(s - 0)] \\ &= -[h_x(x(s)) \cdot x_\Theta(s + 0) + h_\Theta(x(s))/h_x(x(s)) \cdot \dot{x}(s + 0)]. \end{aligned} \tag{45}$$

(iv) *The derivative*,

$$dx(s)/d\Theta = dx(s(\Theta), \Theta)/d\Theta\, |_{\Theta=0}, \tag{46}$$

exists and is given by

$$\begin{aligned} dx(s)/d\Theta &= \dot{x}(s - 0) \cdot s_\Theta + x_\Theta(s - 0) \\ &= \dot{x}(s + 0) \cdot s_\Theta + x_\Theta(s + 0).^{10} \end{aligned} \tag{47}$$

Figure 2 illustrates Lemma 2. It is noted that formulas (41)–(45) are two successive initial-value problems of linear differential equations; a program which solves an initial-value problem of linear differential equations should also be able to solve (41)–(45). For, we may first solve (42) for $x_\Theta(t)$ on (t_0, s) given by (42). From this we may compute $x_\Theta(s - 0)$, and then s_Θ by means of (45). Then, we may proceed to solve (43) for $x_\Theta(t)$ on (s, t_1) given $x_\Theta(s + 0)$ computed from (44).

Equation (47) provides a formula to compute the shift of the switching point $x(s(\Theta), \Theta)$; the first term of the right side expresses its shift arising from the change in the switching time-point $s(\Theta)$, and the second that arising from the change in the function $x(t, \Theta)$.

[10] For a proof of Lemma 2, see the Appendix of the author's discussion paper, "Comparative Dynamics in Optimal Control Theory," Technical Report No. 10, Project on Efficiency of Decision Making in Economic Systems, Harvard University, 1972.

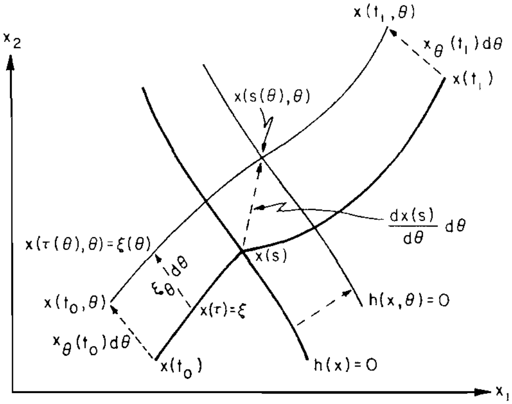

FIG. 2. Shift of the solution of differential equations with discontinuities along a curve $h = 0$.

IV. COMPARATIVE DYNAMICS

In Section II, Pontryagin's condition for the optimum solution $z(t)$ has been reduced into the system (18)–(20). This system is a two-point boundary-value problem of ordinary differential equations possibly with discontinuities in the state variables. In the present section, we make use of Lemma 2 of the previous section to present a set of conditions sufficient for the function $z(t) = z(t, 0)$ to be differentiable with respect to Θ near $\Theta = 0$ and to obtain formulas to compute the derivative.

THEOREM. *Suppose that a unique (regular) optimum solutionz* (t) $(t_0 \leq t \leq t_1)$ *of* (18)–(20) *exists. For simplicity, we state the theorem for the case* $q = 1$; *we write* $s_1 = s$, $h^1 = h$, *etc. The function* $F(z, \Theta)$ *is assumed continuously differentiable for all* (z, Θ) *such that* Θ *is near* 0 *and* z *is regular at* Θ. *The functions* $h(z, \Theta)$ *and* $\Psi^i(z, \Theta)$ *are also continously differentiable in neighborhoods of* $(z(s), 0)$ *and* $(z(t_i), 0)$, *respectively. Suppose further that the following is satisfied by* $z(t)$:

$$h_z(z(s)) \cdot F(z(s \pm 0)) \neq 0. \tag{48}$$

Then, Lemma 2 *implies that there exist functions* $\sigma(\eta, \Theta)$ *and* $y(t, \eta, \Theta)$ *satisfying*

$$\dot{y}(t, \eta, \Theta) = F(y(t, \eta, \Theta), \Theta) \qquad (t_0 \leq t \leq t_1,\ t \neq \sigma(\eta, \Theta)), \tag{49}$$

$$y(t_0, \eta, \Theta) = \eta, \qquad \text{and} \tag{50}$$

$$h(y(\sigma(\eta, \Theta), \eta, \Theta), \Theta) = 0, \tag{51}$$

where y is a $(2n)$-vector (state variables), η a $(2n)$-vector representing the initial value for y (parameters), and Θ a number (a parameter). (This can be shown by applying Lemma 2 $(2n+1)$-times for the $(2n)$ components of η and Θ.) Furthermore, the derivatives

$$\begin{aligned} \sigma_\eta &= (\partial/\partial\eta)\,\sigma(z(t_0), 0), \\ \sigma_\Theta &= (\partial/\partial\Theta)\,\sigma(z(t_0), 0), \end{aligned} \tag{52}$$

$$\begin{aligned} y_\eta(t) &= (\partial/\partial\eta)\,y(t, z(t_0), 0), \qquad \text{and} \\ y_\Theta(t) &= \partial/\partial\Theta\, y(t, z(t_0), 0) \qquad (t \neq s) \end{aligned} \tag{53}$$

exist. We assume that

$$|A| = \begin{vmatrix} A^0 \\ A^1 \end{vmatrix} = \begin{vmatrix} \Psi_z^{\,0} \\ \Psi_z^{\,1} \cdot y_\eta^{\,1} \end{vmatrix} \neq 0, \tag{54}$$

where A is a $(2n, 2n)$-matrix, $|A|$ is the determinant of A, $A^0 = \Psi_z^{\,0}$ and $A^1 = \Psi_z^{\,1} \cdot y_\eta^{\,1}$ are $(n, 2n)$-matrices, $\Psi_z^{\,i} = \Psi_z^{\,i}(z(t_i))$ is the derivative of $\Psi^i(z, \Theta)$ with respect to z at $(z(t_i), 0)$, and $y_\eta^{\,1} = y_\eta(t_1)$ is a $(2n, 2n)$-matrix defined by (53).

Then, there exists a positive number, say $\hat{\Theta}$, for which the following is true:

(i) *For each $\Theta(|\Theta| < \hat{\Theta})$, there uniquely exists a switching time-point $s(\Theta)$ and a unique regular solution $z(t, \Theta)$ satisfying*

$$\dot{z}(t, \Theta) = F(z(t, \Theta), \Theta) \qquad \text{for all } t \neq s(\Theta), \tag{55}$$

$$h(z(s(\Theta), \Theta), \Theta) = 0, \qquad \text{and} \tag{56}$$

$$\Psi^i(z(t_i, \Theta), \Theta) = 0 \qquad (i = 0, 1). \tag{57}$$

(ii) *The derivatives,*

$$s_\Theta = s_\Theta(0) = (d/d\Theta)\, s(\Theta)|_{\Theta=0}, \tag{58}$$

$$z_\Theta(t) = z_\Theta(t, 0) = (\partial/\partial\Theta)\, z(t, \Theta)|_{\Theta=0} \qquad (t \neq s), \qquad \text{and} \tag{59}$$

$$(d/d\Theta)\, z(s) = (d/d\Theta)\, z(s(\Theta), \Theta)|_{\Theta=0}, \tag{60}$$

exist and satisfy

$$\dot{z}_\Theta(t) = F_z(z(t)) \cdot z_\Theta(t) + F_\Theta(z(t)) \qquad (t \neq s), \tag{61}$$

$$z_\Theta(t_0) = -A^{-1} \cdot B, \tag{62}$$

$$z_\Theta(s+0) = z_\Theta(s-0) - [\dot{z}(s+0) - \dot{z}(s-0)]\, s_\Theta, \tag{63}$$

$$s_\Theta = -[h_z(z(s)) \cdot z_\Theta(s \pm 0) + h_\Theta(z(s))/h_z(z(s)) \cdot \dot{z}(s \pm 0)], \quad \text{and} \tag{64}$$

$$(d/d\Theta)\, z(s) = \dot{z}(s \pm 0) \cdot s_\Theta + z_\Theta(s \pm 0), \tag{65}$$

where

$$B = \begin{vmatrix} B^0 \\ B^1 \end{vmatrix} = \begin{vmatrix} \Psi_z^{\ 0} \cdot y_\Theta^{\ 0} + \Psi_\Theta^{\ 0} \\ \Psi_z^{\ 1} \cdot y_\Theta^{\ 1} + \Psi_\Theta^{\ 1} \end{vmatrix} \tag{66}$$

is a $(2n)$*-vector,* $B^i = \Psi_z^{\ i} y^i_\Theta + \Psi_\Theta^{\ i}$ *is an n-vector,* $\Psi_\Theta^{\ i} = \Psi_\Theta^{\ i}(z(t_i))$ *is the derivative of* $\Psi^i(z, \Theta)$ *with respect to* Θ *at* $(z(t_i), 0)$, *and* $y_\Theta^{\ i} = y_\Theta(t_i)$ *is a* $(2n)$*-vector defined by* (53).[11]

Some remarks on this theorem follow. First, the assumption of continuous differentiability of $F(z, \Theta)$ would be satisfied if the derivatives $f_x, f_u, f_\Theta, f_{xx}, f_{xu}, f_{x\Theta}, f_x^{\ 0}, f_u^{\ 0}, f_\Theta^{\ 0}, f_{xx}^{\ 0}, f_{xu}^{\ 0}$, and $f_{x\Theta}^{\ 0}$ exist and are continuous [see (8), (15), and (17)]; we need the second-order derivatives to obtain s_Θ and $z_\Theta(t)$. (In the simple maximization problem, we require the existence of the second-order derivative f_{xx} to obtain x_Θ. See (2).)

Second, (48) and (54) are assumed for regularity (not in the sense defined in Section II but in the general sense). If (48) does not hold, then it is possible that the number of switching time-points changes when Θ varies near 0 or even that the optimum solution $z(t, \Theta)$ is no more regular (in the sense defined in Section II) for a small Θ. If (54) is not satisfied, then it is possible that $z(t, \Theta)$ is no more unique for a small Θ so that the derivative $z_\Theta(t)$ may not be defined.

Third, (61–64) are successive initial-value problems of linear differential equations. They are the formulas for computing the derivatives s_Θ and $z_\Theta(t)$. It is straightforward (if tedious) to write down Eq. (61) in terms of the original notation $f(x, u, \Theta)$, $f^0(x, u, \Theta)$, and $u(p, x, \Theta)$ [see (8), 15), and (17)]. In applications, however, (61) may be obtained simply by differentiating (55) with respect to Θ and putting $\Theta = 0$.

Fourth, the derivative of the maximized objective function with respect to the parameter $(dv/d\Theta)$ may be obtained by substituting $x(t, \Theta)$ and $u(t, \Theta)$, respectively, for $x(t)$ and $u(t)$ in (3) and then differentiating (3) with respect to Θ. Furthermore, $dv/d\Theta$ can be expressed in a simpler and more useful form by using the auxiliary variables $p(t)$.[12]

In the following, we consider two special cases of the theorem by introducing further assumptions on the end and the transversality conditions (20):

(a) The state variable at t_0 is fixed at a point determined solely by the parameter, and the state variable at t_1 is left free:

$$\Psi^0(z, \Theta) = \Psi^0(p, x, \Theta) = x - \zeta(\Theta), \quad \text{and} \quad \Psi^1(z, \Theta) = \Psi^1(p, x, \Theta) = p$$

[11] For a proof of this theorem, see Section IV of the paper referred to in footnote 10.
[12] See Section V of the paper referred to in footnote 10.

The conditions (20) now read

$$x(t_0, \Theta) - \zeta(\Theta) = 0, \qquad p(t_1, \Theta) = 0, \tag{20a}$$

where ζ is an n-vector-valued function of Θ which is smooth near $\Theta = 0$. Let us put

$$z_\Theta(t) = (p_\Theta^*(t)\, x_\Theta^*(t)) \qquad (t \neq s), \tag{59a}$$

$$\eta = (\pi, \xi), \qquad \text{and} \tag{50a}$$

$$y_\eta(t) = \begin{vmatrix} p_\pi(t) & p_\xi(t) \\ x_\pi(t) & x_\xi(t) \end{vmatrix}, \qquad y_\Theta(t) = (p_\Theta(t)\, x_\Theta(t)), \quad (t \neq s). \tag{53a}$$

We then obtain

$$A = \begin{vmatrix} p_\pi(t_1) & p_\xi(t_1) \\ 0 & 0 \end{vmatrix}, \qquad B = \begin{vmatrix} p_\Theta(t_1) \\ -\zeta_\Theta \end{vmatrix},$$

so that the regularity condition (54) and the initial condition (62) for this case can be written as

$$| p_\pi(t_1)| \neq 0, \tag{54a}$$

$$p_\Theta^*(t_0) = -p_\pi(t_1)^{-1} \cdot (p_\xi(t_1)\, \zeta_\Theta + p_\Theta(t_1)), \qquad x_\Theta^*(t_0) = \zeta_\Theta, \tag{62a}$$

where ζ_Θ is the derivative of $\zeta(\Theta)$ at $\Theta = 0$.

(b) The initial and the terminal values of the state variables are determined solely by the parameter:

$$\Psi^i(z, \Theta) = \Psi^i(p, x, \Theta) = x - \zeta^i(\Theta) \qquad (i = 0, 1),$$

say. For this case, we obtain

$$A = \begin{vmatrix} 0 & I \\ x_\pi(t_1) & x_\xi(t_1) \end{vmatrix}, \qquad B = \begin{vmatrix} -\zeta_\Theta \\ x_\Theta(t_1) - \zeta_\Theta^1 \end{vmatrix},$$

and hence

$$| x_\pi(t_1)| \neq 0, \tag{54b}$$

$$p_\Theta^*(t_0) = -x_\pi(t_1)^{-1} \cdot (x_\xi(t_1)\, \zeta_\Theta^0 + x_\Theta(t_1) - \zeta_\Theta^1), \qquad x_\Theta^*(t_0) = \zeta_\Theta^0. \tag{62b}$$

V. An Example—Comparative Dynamics in a Model of Optimal Capital Accumulation

In the present section, we show how the comparative dynamics as presented in the preceding sections can be applied to economic problems. A standard model of optimal capital accumulation developed by Cass [3] will be used as an example.[13] For short, we follow his notation and do not reproduce the definition of the variables used in [3].

To begin with, we write Pontryagin's condition for Cass's model:[14]

$$\dot{q} = (\delta - f'(k))\, q, \qquad \dot{k} = f(k) - c(q), \tag{18c}$$

$$k(0) = k^0, \qquad k(T) = k^T, \tag{20c}$$

where $u'(c(q)) = q$. For simplicity, it is assumed that $\lambda = 0$ and that $z > 0$ throughout the planning period $(0, T)$ (i.e., no switching).

We consider the discount rate δ as the parameter. We then write: $q = q(t, \delta)$, $k = k(t, \delta)$, $c = c(q(t, \delta)) = c(t, \delta)$, etc. It is seen from (62b) that we need to obtain $k_{q^0}(T)$ and $k_\delta(T)$, where $q(0) = q^0$. To do this, let us differentiate (18c) and (20c) with respect to q^0, obtaining

$$\begin{vmatrix} \dot{q}_{q^0} \\ \dot{k}_{q^0} \end{vmatrix} = \begin{vmatrix} \delta - f' & -f''q \\ -c' & f' \end{vmatrix} \cdot \begin{vmatrix} q_{q^0} \\ k_{q^0} \end{vmatrix},$$

$$q_{q^0}(0) = 1, \qquad k_{q^0}(0) = 0.$$

The sign of each element in the square matrix above is determined from Cass's assumption,[15]

$$\begin{vmatrix} \dot{q}_{q^0} \\ \dot{k}_{q^0} \end{vmatrix} = \begin{vmatrix} ? & + \\ + & + \end{vmatrix} \cdot \begin{vmatrix} q_{q^0} \\ k_{q^0} \end{vmatrix}.$$

To investigate the path of $q_{q^0}(t)$ and $k_{q^0}(t)$, we construct a phase diagram (see Fig. 3). Each arrow in the phase diagram indicates a possible direction of the path. Since the path starts at a point on the upper half of the vertical axis, it is clear from the diagram that

$$q_{q^0}(t) > 0, \qquad k_{q^0}(t) > 0 \qquad \text{for all } t. \tag{67}$$

[13] The method presented in this section was once introduced by Oniki [7].

[14] See Cass [3, pp. 837–838].

[15] See Cass [3, pp. 834–835].

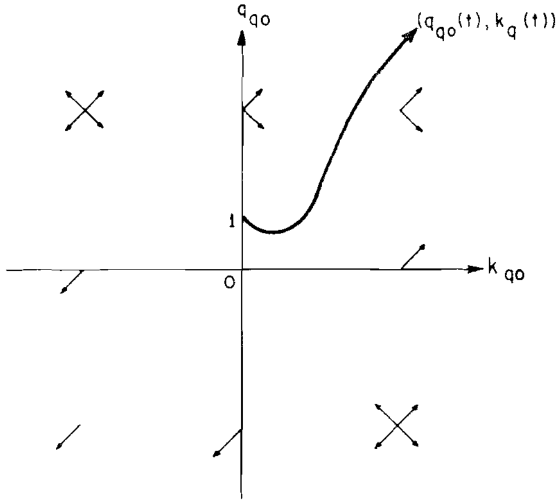

FIG. 3. Phase-diagram of $(q_{q0}(t), k_{q0}(t))$.

Following a similar method, we obtain

$$\begin{aligned} \begin{vmatrix} \dot{q}_\delta \\ \dot{k}_\delta \end{vmatrix} &= \begin{vmatrix} \delta - f' & -f''q \\ -c' & f' \end{vmatrix} \begin{vmatrix} q_\delta \\ k_\delta \end{vmatrix} + \begin{vmatrix} q \\ 0 \end{vmatrix} \\ &= \begin{vmatrix} ? & + \\ + & - \end{vmatrix} \begin{vmatrix} q_\delta \\ k_\delta \end{vmatrix} + \begin{vmatrix} + \\ 0 \end{vmatrix}; \\ q_\delta(0) &= 0, \qquad k_\delta(0) = 0. \end{aligned} \tag{68}$$

From this and a phase diagram (see path A in Fig. 4), we conclude that

$$q_\delta(t) > 0, \qquad k_\delta(t) > 0 \qquad \text{for all } t. \tag{69}$$

Then, (62b, 67, and 69) yield

$$\begin{aligned} q_\delta{}^*(0) &= -k_\delta(T)/k_{q0}(T) < 0, \\ k_\delta{}^*(0) &= 0. \end{aligned} \tag{62c}$$

In addition, since $k^*(T, \delta) = k^T$,

$$k_\delta{}^*(T) = 0. \tag{70}$$

We finally consider variational equations for $q_\delta{}^*(t)$ and $k_\delta{}^*(t)$, which is a special case of (61). In fact, these equations are obtained by substituting $q_\delta{}^*$ and $k_\delta{}^*$ into (68); we may use Fig. 4 to describe the path of $q_\delta{}^*(t)$,

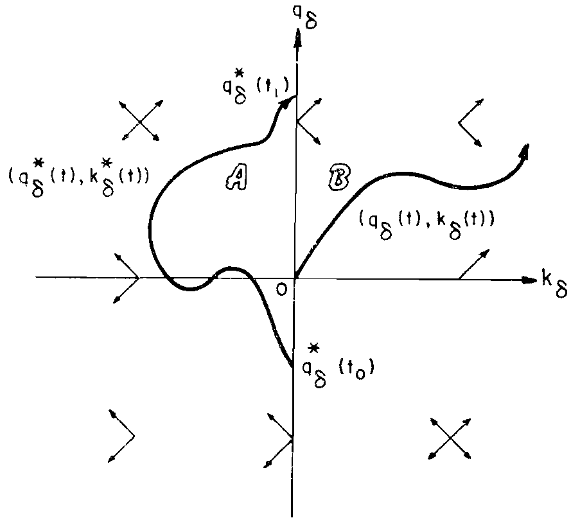

FIG. 4. Phase-diagrams of $(q_\delta(t), k_\delta(t))$ and $(q_\delta^*(t), k_\delta^*(t))$.

$k_\delta^*(t)$. In veiw of the end conditions (62c) and (70), we know that the path must start at a point on the lower half of the vertical axis, and must terminate at a point on the upper half of it. Hence, it stays within the second and the third quadrants. It is seen that the path may cross the horizontal axis more than once, bot never the vertical axis. Path *B* in Fig. 4 is a typical one.

From the above consideration, we can conclude that

$$\begin{aligned} &k_\delta^*(t) < 0, \qquad \text{for all } t, \\ &c_\delta^*(t) \quad c'(q(t)) \cdot q_\delta^*(t) > 0 \text{ for small } t, \\ &\qquad\qquad\qquad\qquad < 0 \text{ for large } t, \end{aligned} \tag{71}$$

which summarizes the effect of a change in δ on the optimum path.

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