THE COST OF COMMUNICATION IN ECONOMIC ORGANIZATION*

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I. INTRODUCTION AND BACKGROUND

The purpose of this paper is to propose a method for measuring the cost of communication in alternative economic systems allocating scarce resources. An economic system, particularly if large and complex, needs some internal communication in order to find desirable allocations, since relevant economic knowledge is dispersed among its members (individual agents). Information transmitted for this purpose may be of various kinds and differs among different systems. It is possible, however, to measure the amount of transmitted information by using a certain unit and to compare the cost of communication among different systems. To do this, we use certain results obtained in communication theory.[1] In this paper we first introduce a method for approximating economic data in order to make it possible to apply communication theory to our problem. We then compute the cost of communication in economic systems using a price mechanism and the cost in systems under centralized control, limiting our attention to a simple problem of resource allocation.

The cost of communication is one element of *transaction or organizational costs* (costs of running economic systems), as distinct from "real" or *substantive costs* (costs of production, transportation, etc.)[2] The need for a theory of transaction costs arises from the fact that the

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[1] The foundation for communication theory was laid by Shannon. See C. E. Shannon, "The Mathematical Theory of Communication," *Bell System Technical Journal*, XXVII (1948), 379-423, 623-56. Reprinted in C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication* (Urbana: University of Illinois Press, 1949).

[2] See K. J. Arrow, "The Organization of Economic Activity: Issues Pertinent to the Choice of Market versus Nonmarket Allocation," in *The Analysis and Evaluation of Public Expenditures: The PPB System,* U. S. Congress,

overall efficiency and viability of the economic system depends not only on its performance but also on the amount of resources devoted to its operation.[3] The existence of transaction costs has long been recognized, if only from limited studies relating to particular problems.[4] The fact that transaction costs must exist with any mode of allocation was recently stressed by Arrow and Hurwicz.[5]

An important subclass of transaction costs is informational costs of decision making, of which a comprehensive investigation was made by J. Marschak.[6] He pointed out that the process of decision making could be regarded as a sequence of transformations of one type of information into another; he then identified, among others, three major "transformers": inquiring (data collection), communicating (message transmission), and deciding (computation). Informational costs may be attached to each of these transformers.

In this paper we limit our attention to the cost of communication because we can compute it by using communication theory, while no such theory is yet available for computing other informational costs. We deal with systems of two firms solving a simple problem of resource allocation. Suppose that each firm possesses technology for producing an output commodity from an input commodity and that the total supply of the input commodity is fixed. The problem is to allocate the input commodity between the two firms efficiently, that is, to maximize the total output. This, of course, is done by equating marginal products between the firms. However, suppose that each firm knows its own technology but nothing of the other firm's technology or of the total supply of input. Suppose further that there is a *central agent* who knows the total amount of the input but not the technology of the firms. The problem is then how the firms and the central agent coordinate themselves to find efficient allocations. The problem of economic organization here is to devise a set of behavior rules for the three

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Joint Economic Committee, printed by the U. S. Government Printing Office, 1969, vol. I, pp. 47-64. Arrow states: "The distinction between transaction costs and production costs is that the former can be varied by a change in the mode of resource allocation, while the latter depend only on the technology and tastes, and would be the same in all economic systems" (p. 60).

3 See L. Hurwicz, "On Informationally Decentralized Systems," in R. Radner and B. McGuire, eds., *Decision and Organization* (Amsterdam: North-Holland Publishing Co., 1972), Ch. 14.

4 For instance, Lange and Taylor's well-known price system was based upon the recognition that the cost of solving "millions of equations" by the Central Planning Board, a cost of running a centralized economy, would be prohibitively high. (O. Lange and F. M. Taylor, *On the Economic Theory of Socialism;* Minneapolis: University of Minnesota Press, 1938, reprinted by McGraw-Hill Book Co., 1964.) F. H. Knight attributed the existence of the firm and of profit to the presence of uncertainty, thus recognizing the cost of obtaining information. (F. H. Knight, *Risk, Uncertainty and Profit;* Boston: Houghton-Mifflin Co., 1921), Ch. 9, reprinted by London School of Economics and Political Science, 1948.) Furthermore, Coase and Demsetz emphasized the existence of transaction costs accompanying the provision of a market for (or governmental control on) external diseconomies. (R. H. Coase, "The Problem of Social Cost," *Journal of Law and Economics,* III (Oct. 1960), 1-40. H. Demsetz, "The Exchange and Enforcement of Property Rights," *Journal of Law and Economics,* VII (Oct. 1964), 11-26.)

5 See Arrow, *op. cit.,* pp. 59-60; and Hurwicz, *op. cit.,* Section I.

6 J. Marschak, "Economics of Inquiring, Communicating, Deciding," *American Economic Review,* LVIII (May 1968), 1-18; and "Economics of Information Systems," in M. D. Intriligator, ed., *Frontiers of Quantitative Economics* (Amsterdam: North-Holland Publishing Co., 1971), Ch. 2. A sketch of informational activities was also given by T. Marschak, "Computation in Organizations: Comparison of Price Mechanisms and Other Adjustment Processes," in K. Borch and J. Mossin, eds., *Risk and Uncertainty* (New York: Macmillan, 1968), pp. 311-49.

agents such that by following the rules they may find efficient allocations.[7] We shall study two systems. The first is a *centralized system* in which each firm informs the central agent what its technology is, once and for all, and lets the latter do necessary calculations. The second is the ordinary *price mechanism* that follows the law of supply and demand; the central agent is an auctioneer and adjusts prices until equilibrium is reached. In each system *economic data* are transmitted between the central agent and the firms. The data may consist of prices, quantities of a commodity, or knowledge about production technology. The cost of communication is attached to each transmission of economic data.

Major difficulty in measuring the cost of communication arises from the fact that the cost depends on the accuracy of communication. (This, indeed, is true not only for communication but also for informational activities in general.) Of course, if the accuracy is higher, or equivalently, if errors are smaller, the cost will be greater. The accuracy of decision on resource allocation is also affected by the accuracy of communication: the former depends on the accuracy of messages received, which in turn depends on the accuracy of messages sent and on the accuracy of the channel transmitting messages. It seems almost impossible to investigate the relations among all these factors in full generality.[8] Here we consider the problem with the following simplification. First, we abstract from specifying a particular channel by using a major result in communication theory: the amount of information to be transmitted may consistently be defined independently of channels.[9] Second, we introduce certain methods for approximating economic data and assume that the approximate data thus introduced commonly be used for communication and decision making. The degree of accuracy for this approximation is treated as an exogenously given parameter.[10] Such simplification together with certain additional assumptions makes it possible to compute the cost of communication explicitly and to compare alternative economic systems with respect to the cost.

Our results suggest that the relative magnitude of the cost of communication in the two systems depends on the accuracy of approximation. If high accuracy is required, then the cost of communication in the price mechanism will be much lower than the cost in the centralized system; this is an expected result. On the other hand, however, if one is satisfied with rather low accuracy, then the reverse may be the case.

Since the scope of the paper is admittedly limited, we do not intend to provide any overall evaluation of the two systems. However, we do intend to present methods that might be useful for studying informational costs of decision making. For instance, the way in which we approximate economic data may be applicable to analyzing “information transformers” other

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[7] See Hurwicz, *op. cit.*, for more general formulation of the problem.

[8] However, see J. Marschak, “Economics of Information Systems.”

[9] This is implied by Shannon’s coding theorems in noisy and noiseless channels. See Shannon, *op. cit.* Also see Appendix A below.

[10] See Section II.

than communication.

The problem of economic organization, of which this paper deals with a special subject, has been investigated from a number of different aspects. Arrow suggested the existence of problems central to theoretical studies --- internal and external uncertainty in large organizations, joint decision making, learning in organizations, not to mention others.[11] Here we have to assume away these problems except that of internal uncertainty.

An important issue in the theory of economic organization has been the choice between centralized and decentralized modes of decision making. The well-known controversy on socialist's planning resulted in almost unanimous preference for decentralized decision making by individual agents coordinated through a price mechanism. The reason for the preference was that the cost of communication and that of computation in a centralized economy would be prohibitively high. This is partly because the number of economic variables and relations is large and partly because relevant information is difficult to transmit.[12]

Since that time the problem of centralization and decentralization has been considered mostly from informational aspects. J. Marschak and Radner constructed the theory of teams, emphasizing various information structures of teams.[13] They computed the cost of communication and that of observation in certain teams without using communication theory.[14] In a previous work, however, Marschak presented a simple model with which the cost of communication is measured by using communication theory.[15]

T. Marschak investigated the problem of centralization and decentralization systematically.[16] He assumed that production technology was expressed by a set of linear activities and computed the cost of communication and that of computation in both centralized and decentralized systems. The unit of communication cost was the transmission of a real number, and the unit of computation cost was an arithmetic operation. The following are the major differences between Marschak's model and ours: (1) Marschak considered both the cost of communication and that of computation, while we measure the cost of communication only. (2) He assumed that production technology was expressed by linear activities, while we do not.

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11 K. J. Arrow, "Control in Large Organizations," *Management Science*, X (1963-1964), 397-408. Reprinted in K. J. Arrow, *Essays in the Theory of Risk-Bearing* (Chicago: Markham Publishing Co., 1971). Also, K. J. Arrow, "Research in Management Controls," in C. P. Bonini, R. K. Jaedicke, and H. M. Wagner, eds., *Management Controls: New Directions in Basic Research* (New York: McGraw-Hill, 1964), Ch. 17.

12 See, e.g., Lange and Taylor, *op. cit.*; and F. A. Hayek, "The Use of Knowledge in Society," *American Economic Review*, XXXV (Sept. 1945), 520-30.

13 J. Marschak and R. Radner, *Economic Theory of Teams* (New Haven: Yale University Press, 1972); also, J. Marschak, "Towards an Economic Theory of Organization and Information," in R. M. Thrall, C. H. Coombs, and R. L. Davis, eds., *Decision Processes* (New York: John Wiley, 1954), Ch. 14; and "Problems in Information Economics," in Bonini, Jaedicke, and Wagner *op. cit.*, Ch. 3; R. Radner, "The Evaluation of Information in Organization," in J. Neyman, ed., *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley: University of California Press, 1961), Vol. I, pp. 491-530.

14 Marschak and Radner, *op. cit.*, Chs. 4 and 9.

15 J. Marschak, "Towards an Economic Theory of Organization and Information," Section 5.

16 T. Marschak, "Centralization and Decentralization in Economic Organizations," *Econometrica*, XXVII (July 1959), 399-430. See also T. Marschak, "Computation in Organizations."

(3) We use Shannon's measure of information to express the cost of communication, while he measured it by the number of real numbers transmitted. (4) Both Marschak and we explicitly consider the errors (inaccuracy) of allocations. The two formulations, however, differ significantly. We introduce approximation of economic data at the outset, thus making the choice of the accuracy level entirely exogenous to the analysis. Marschak assumed that, while the adjustment was proceeding, the previous (non-optimal) allocation was in effect, so that higher accuracy could be obtained by delaying the time to adopt the optimal allocation.

Hurwicz formulated models of resource allocation and adjustments so general that centralized and decentralized systems are but special cases.[17] He stressed informational decentralization: initial dispersion of information among individual agents and costliness of channels transmitting information. A channel is consistent with informational decentralization if its capacity is such that economic data that are finite-dimensional vectors (like prices, commodities) can be transmitted but those that are infinite-dimensional vectors (like production functions and utility functions) cannot. The main difference between Hurwicz's formulation and ours is that we measure the cost of communication directly by introducing approximation of economic data. His definition of informational decentralization seems to correspond to one of our results that the cost of communication in centralized systems is much higher than that in decentralized price mechanisms if the required level of accuracy is high.

II. THE MODEL AND THE APPROXIMATION OF ECONOMIC DATA

In this section we formulate a simple problem of resource allocation into a model with respect to which the cost of communication is computed. First, we state the model in ordinary terminology. Next, we introduce a method for approximating economic data that appear in the model formulated and restate the model in terms of the approximate data. The revised model thus obtained is suitable for computing the cost of communication.

A. The Original Model

Let us consider two firms: $i = 1,2$. Each firm possesses technology represented by a production function,

(1) $$y_i = f_i(x_i) \qquad (i = 1,2),$$

where x_i is the input commodity and y_i the output commodity. It will be assumed that the production function f_i is continuous, non-decreasing, and concave.

[17] L. Hurwicz, "Optimality and Informational Efficiency in Resource Allocation Processes," in K. J. Arrow, S. Karlin, and P. Suppes, eds., *Mathematical Methods in the Social Sciences* (Stanford: Stanford University Press, 1960), pp. 27-46. Reprinted in K. J. Arrow and T. Scitovsky, eds., *Readings in Welfare Economics* (Homewood: Richard Irwin, 1969), pp. 61-80. Also, L. Hurwicz, "On Informationally Decentralized Systems;" "Centralization and Decentralization in Economic Processes," in A. Eckstein, ed., *Comparison of Economic Systems: Theoretical and Methodological Approaches* (Berkeley: University of California Press, 1971), Ch. 3; and "On the Concept and Possibility of Informational Decentralization," *American Economic Review,* LIX (May 1969), 513-24.

We next assume that the total supply of the input commodity is fixed at $x^0 > 0$. The objective of the firms is to find an allocation (x_1, x_2) that maximizes the total output,

(2) $y = y_1 + y_2 = f_1(x_1) + f_2(x_2)$,

subject to the input constraints

(3) $x_1 + x_2 \leq x^0, \quad x_i \geqq 0 \qquad (i = 1,2)$.

B. The Approximation

The formulation stated above is not suitable for our purpose, since it is difficult to express the degree of accuracy with it. We introduce approximation in the following way.

Consider the input commodity, its quantity being denoted by x. We assume that there is an upper bound of x such that no quantity of x beyond this bound will ever be considered. For simplicity, we take the total supply x^0 to be this upper bound. (Note that it is safe to do so.) We may then limit our attention to the interval,

(4) $X = \{x; 0 \leqq x \leqq x^0\}$.

Furthermore, we assume that it is not interesting to know a precise quantity of x, but that it is sufficient to have an approximate value only. We adjust the unit of x so that $x^0 = m$, where m is a given positive integer. We then assume that only integer values of x are considered. Define the set of the integer values of x:

(5) $M = \{1,2,\dots,m\}, \qquad M_0 = M \cup \{0\}$.

The choice of m determines the degree of accuracy in this approximation.[18]

Let p be the price of the input commodity in terms of the output commodity. We assume that p *is* approximated in the same way as x:

(6) $P = \{p; 0 \leqq p \leqq p^0\}$,

$N = \{1,2,\dots,n\}, \qquad N_0 = N \cup \{0\}$,

where $p^0 > 0$ is the upper bound of p, n is a positive integer chosen exogenously, and the unit of the output commodity is adjusted so that $p^0 = n$. The output commodity y is approximated likewise:

$Y = \{y; 0 < y \leqq y^0 \equiv p^0 x^0 + 1\}$,

(7) $L = \{1,2,\dots,mn+1\}$.

The choice of the upper bound y^0 is made for analytical convenience.[19]

Let us next consider approximating production functions. First of all, we assume that the class of production functions that possibly arise in the allocation problem can be expressed by

(8) $F = \{f: X \rightarrow Y;$

[18] For the present purpose it is not necessary that the distance between two adjacent points in M is constant; all that is needed is that M be a finite set.

[19] The construction above implies that the output commodity is more finely approximated than the input commodity and the price; there is asymmetry between x and y. The construction, however, may be natural, since x and p are treated symmetrically so that the approximation in the dual space may be obtained immediately from the present approximation.

f is continuous, non-decreasing, and concave;

$$0 < f(0) \leqq 1, \;\; 0 \leqq f'(0) \leqq p^0\},$$

where $f'(0)$ denotes the right derivative of f at 0. Figure I illustrates the range of the graphs of production functions belonging to F. Among the conditions that characterize the set F, the first three are standard in economic theory, calling for no further comments. The last two conditions, stated in terms of inequalities, are imposed chiefly for analytical convenience.[20]

We shall "approximate" the set F of production functions with a finite set, as we approximated the sets X, P, and Y, respectively, with the sets M, N, and L, all being finite sets. We do this by considering the values of a production function $f \in F$ on M and then approximating those values with points in L. First, let us choose $g(j) = g_f(j) \in L$ such that

(9) $\quad g(j) - 1 < f(j) \leqq g(j), \qquad j \in M;$

i.e., $g(j)$ is the integer approximating $f(j)$. The subscript f attached to g reminds us that g is obtained from f; the subscript may be omitted if no ambiguity arises. Define

(10) $\quad h(j) = h_f(j) = h_g(j) = g(j) - g(j-1), \qquad j \in M.$

(We put $g(0) = 1$.) Immediately we have

(11) $\quad g(j) = g(0) + \sum_{j'=1}^{j} h(j'), \qquad j \in M.$

Inequality (9) shows that the function $g = g_f$ approximates the original production function f on M. We call g_f the *approximate production function* of f. From definition (10) it is expected that the function h approximates the slopes of f. In fact, it can be shown that

(12) $\quad 0 \leqq h(j) \leqq n,$ i.e., $h(j) \in N_0, \qquad j \in M;$

(13) $\quad h(j) - 1 < f(j) - f(j-1) < h(j) + 1, \qquad j \in M.$

(Proof is given in Appendix B, Part B.1.) As might be suggested by (13), it is not true that the function $h(j)$, unlike the slope $f(j) - f(j-1)$, is non-increasing in j. We write, for simplicity,

(14) $\quad g = g_f = (g(1), \dots, g(m)) \in L^m,$

$\qquad\quad h = h_f = h_g = (h(1), \dots, h(m)) \in N_0^m.$

The set of approximate production functions is defined by

(15) $\quad G = G_F = \{g_f; f \in F\} \subset L^m.$

The fact that $g = g_f$ approximates f can be shown more precisely. To do this, we extend g, which has been defined only on M, to the set X:

(16) $\quad g(x) = g([x]) + h([x]+1)(x - [x]), \qquad x \in X,$

where $[x] \in M_0$ is the greatest integer not exceeding x. The function $g(x)$ thus extended is a piecewise linear function; its graph is obtained by joining $(m+1)$ points $(j, g(j))$ successively

[20] The condition $0 < f(0) \leqq 1$ could easily be replaced by a more general condition, say, $0 < f(0) \leqq z$, where z is a positive integer. (We should then replace $y^0 = p^0x^0 + 1$ by $y^0 = p^0x^0 + z$ for the obvious reason.) This formulation will be desirable if the origin $x = 0$ is not the point at which physical quantity of x is zero but a lower bound of x.

by segments. It is noted that $g(x)$ may not be concave. It can be shown that [21]

(17) $|g(x) - f(x)| < max[n/4,\ 1], \qquad x \in X.$

That is to say, if n is large, the error of the approximation does not exceed $n/4(= (y^0 - 1)/4m)$. This implies that the error can be made arbitrarily small if a sufficiently large m is chosen. Approximation of f in Figure I with $g = g_f$ is depicted in Figure II.

In the preceding paragraphs we introduced approximate economic data for the input and the output commodities, the price of the input commodity, and the production technology. We now turn to stating the demand function for the input commodity x in terms of the approximate data. Let us first define

(18) $A_0 = A_0(\pi) = A_0(\pi, h)$

$= \{j \in M_0;\ \ h(j') > \pi$ for all $j' \in M$ such that $j' \leqq j\}$,

$A_0 = A_0(\pi) = A_0(\pi, h)$

$= \{j \in M_0;\ \ h(j') < \pi$ for all $j' \in M$ such that $j' > j\}$;

(19) $\xi_0 = \xi_0(\pi) = \xi_0(\pi, h) = max\, A_0,$

$\xi^0 = \xi^0(\pi) = \xi^0(\pi, h) = min\, A_0,$

where $\pi \in N_0$, $h = h_f$, and $f \in F$.

We show that the functions $\xi_0(\pi)$ and $\xi^0(\pi)$ may be interpreted to be the minimum and the maximum, respectively, of the "demand correspondence" at π for the input commodity approximated (in other words, these functions in a sense are the "inverse" of h):[22] for any $\pi \in N_0$,

(20) $0 < j \leqq \xi_0(\pi)$ implies $\pi + 1 \leqq h(j)$;

$\xi_0(\pi) < j \leqq \xi^0(\pi)$ implies $\pi - 1 \leqq h(j) \leqq \pi + 1$;

$\xi^0(\pi) < j \leqq m$ implies $h(j) \leqq \pi - 1$.

(For proof see Appendix B, Part B.3.) For simplicity, we call $\xi_0(\pi)$ and $\xi^0(\pi)$ the *approximate demand functions* for the input commodity. It can also be shown that the demand correspondence formed by ξ_0 and ξ^0 is "continuous" and "non-decreasing" (proved in Appendix B, Part B.2):

(21) $\xi^0(\pi - 1) \geqq \xi^0(\pi) \geqq \xi_0(\pi - 1) \geqq \xi_0(\pi), \qquad \pi \in N,$

(22) $\xi^0(0) = m$ and $\xi_0(n) = 0$.

Figure III illustrates (21).

[21] For proof see the author's discussion paper, "The Cost of Communication in Economic Organization," Project on Efficiency of Decision Making in Economic Systems, Harvard University, Technical Report No. 11, 1972, Appendix II.

[22] Equations (18) through (20) are complicated because the function $h(j)$, which represents approximate marginal products, is not monotone in *j* (although it is either non-increasing or increasing at most by 1, as shown in Appendix B, Part B.4). If h were non-increasing, then we would put $\xi_0 = min\{j; h(j) = \pi\} - 1$ and $\xi^0 = max\{j; h(j) = \pi\}$. Furthermore, if h were strictly decreasing, then we would put simply $\xi_0 = \xi^0 = h^{-1}$.

C. The Model in Terms of the Approximate Data

The allocation problem formulated previously as (2) and (3) can now be restated in terms of the approximate data that we have just introduced. Suppose that information relevant to the allocation problem is dispersed among the three economic agents: f_i is known to firm i only, and x^0 to the central agent only. The objective of an economic system is then to find a set of allocations, say $A = \{(x_1, x_2)\}$, such that

(3) $\quad x_1 + x_2 \leqq x^0, \qquad x_i \geqq 0 \qquad (i = 1,2)$, and

(23) $\quad |f_1(x_1) + f_2(x_2) - y^*| < d$, for all $(x_1, x_2) \in A$,

where $y^*$is the ("true") maximum output in the original allocation problem and d is a bound of allocation errors.

It is seen that an economic system is characterized by an algorithm for finding a set A. Informational activities of each agent and, hence, informational costs and allocation errors will be determined once an algorithm is specified. A general approach to the problem of comparing alternative systems would consider the tradeoff between informational costs and allocation errors. However, here we do not adopt such a general approach but limit our attention to considering special problems only: For the centralized system we compute the cost of communication. For the decentralized price mechanism we compute the cost of communication and estimate allocation errors. All of these tasks will be done in the following section.

III. THE COST OF COMMUNICATION

The present section is devoted to computing the cost of communication in the two economic systems. The discussion of each system is composed of two parts: (1) to describe the system's informational activities to solve the allocation problem approximately and, especially, to describe communication among the three agents; (2) to compute the cost of communication by using Shannon's measure of information.

A. Centralized System

1. The centralized system to be considered here is like that discussed at the early stage of the socialist-planning controversy. The central agent may be the State Government or the Central Planning Board. It first collects from each firm all the information necessary to compute efficient allocations. After computation the central agent informs each firm of an efficient allocation. In terms of the approximate data introduced in the preceding section, the communication may be stated as follows: Each firm sends its approximate production function $g_i = g_{f_i}$ to the central agent. The central agent computes a set A of approximately efficient allocations by using the information given by g_1, g_2, and $m(= x^0)$. It then chooses from A an approximately efficient allocation, say $(j_1, j_2) \in M^2$ and sends it back to the firms.

2. The cost of communication in this system arises from two sources: transmission of (g_1, g_2) and transmission of (j_1, j_2).[23] The sets of possible messages are G and M, respectively, for the two sources. As summarized in Appendix A, $log|G|$ (where $|G|$ denotes the number of elements in G) approximates the minimum number of bits needed to transmit g_i if one does not know the probability with which $g_i \in G$ arises so that a code of equal length is assigned to each element of G. We assume this and use $log|G|$ for the index of the cost of transmitting g_i. Likewise, the cost of transmitting j_i is expressed by $log|M|$.

It is difficult to obtain a simple formula with which $|G|$ can be computed; hence, we consider estimating $|G|$ indirectly. Let

(24) $$G_* = \{g \in L^m;\ h_g(j-1) \geqq h_g(j),$$

$$\text{for } j = 2, \dots, m, \text{ and } h_g \in N_0^m\}$$

$$G^* = \{g \in L^m;\ h_g(j-1) + 1 \geqq h_g(j),$$

$$\text{for } j = 2, \dots, m, \text{ and } h_g \in N_0^m\}.$$

An element of G_*, if extended to X according to formula (16), is a piecewise linear non-increasing concave function from X into Y. Also, an element of G^*, if extended to X likewise, is a piecewise linear continuous function from X into Y with slopes either non-increasing or increasing at most by one. It can be shown that (see Appendix B, Parts B.4 and B.5 for proof)

(25) $$G_* \subset G \subset G^*$$

(26) $$|G_*| = \binom{n+m}{m}, \qquad |G^*| = \binom{n+2m-1}{m}.$$

We then obtain, making use of the formulas, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ and $a! \approx a^{a+1/2} e^{-a} \sqrt{2\pi}$, the following equations:

(27) $$log|G_*| = log \frac{(m+n)^{m+n+1/2}}{m^{m+1/2}\ n^{n+1/2}} + r,$$

(28) $$log|G^*| = log \frac{(2m+n-1)^{2m+n-1/2}}{m^{m+1/2}\ (m+n-1)^{m+n-1/2}} + r,$$

where $r = -\,log\sqrt{2\pi}$. If, in particular, $m = n$, then (see Appendix B, Part B.6)

(29) $$log|G_*| = m(log\, 4 + o(m)) + r,$$

$$log|G^*| = m(log\, 27/4 + o(m)) + r,$$

where $lim_{k\to\infty}\, o(k) = 0$. Since (25) implies that $|G_*| \leqq |G| \leqq |G^*|$, we can assert that $log|G|$

[23] It was the cost of transmitting the g_i's that was considered prohibitively high by most of the participants in the planning controversy; centralized decision making for a whole society was not considered feasible. Such centralization of information, however, exists in the real world on much smaller scales, as seen in modern business, government, and other organizations. This means that the transmission of information like production technology may pay sometimes or at least that it is not always prohibitively costly.

increases approximately linearly if $m = n$ and m is large.[24]

The cost of transmitting an approximately optimal allocation from the central agent to a firm is easily calculated; it is equal to $log|M| = log\, m$.

The total amount of information transmitted within the centralized system is (note that there are two firms)

(30) $\quad C_1 = 2\, log|G| + 2\, log|M|$.

In view of (29), we obtain (if $m = n$)

(31) $\quad 2\{m(log\, 4 + 0(m)) + r + log\, m\} \leqq C_1 \leqq 2\{m(log\, 27/4 + 0(m)) + r + log\, m\}$.

B. Price Mechanism

1. This system may be a competitive market that follows the law of supply and demand; the central agent is the auctioneer of the market. Or, it may be socialist's price system, where the Central Planning Board adjusts prices according to the Lange-Taylor scheme. Adjustments in this system proceed as follows: Given a current price, each firm computes the demand and then reveals it to the central agent, who then computes the aggregate excess demand. The central agent raises (lowers) the price if the excess demand is positive (negative). The process is continued until equilibrium is reached.

For the problem we are dealing with, this process may be stated in terms of the approximate data in the following way: First, given a current approximate price $\pi \in N_0$, firm i responds with the approximate demand functions $\xi_{0i}(\pi_0) = \xi_{0i}(\pi, h_i)$ and ${\xi_i}^0(\pi) = {\xi_i}^0(\pi, h_i)$, where $h_i = h_{f_i}$ $(i = 1,2)$. Then, the central agent computes the approximate aggregate demand. If the equilibrium condition,

(32) $\quad \xi_{01}(\pi) + \xi_{02}(\pi) \leqq m \leqq {\xi_1}^0(\pi) + {\xi_2}^0(\pi)$,

is satisfied, the process is terminated.[25] Otherwise, the central agent adjusts the current price π following the law of supply and demand and chooses a new price from N_0.

We first consider the equilibrium condition (32). Let π^* denote an approximate price satisfying (32). From (21) and (22) it can easily be derived that such π^* always exists. Furthermore, a bound of allocation errors at π^* can be estimated:[26]

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[24] The result obtained here is at least not inconsistent with the assertion in the preceding note. Since the number of bits needed to transmit $g_i(= log|G|)$ is approximately *linear* in m, the transmission cost may not be prohibitively high even for large m.

[25] That (32) is an equilibrium condition in terms of the approximate data may be seen from the following: Choose (j_1, j_2) in such a way that $j_1 + j_2 = m$ and $\xi_{0i} \leqq j_i \leqq {\xi_i}^0$ $(i = 1,2)$, where $\xi_{0i} = \xi_i(\pi)$, ${\xi_i}^0 = {\xi_i}^0(\pi)$, and π satisfy (32). Then it follows from (20) that (i) $h_i(j_i) \geqq \pi + 1$ and $h_i(j_i + 1) \leqq \pi - 1$, if $j_i = \xi_{0i} = {\xi_i}^0$, and that (ii) $\pi - 1 \leqq h_i(j_i) \leqq \pi + 1$ if $\xi_{0i} < j_i \leqq {\xi_i}^0$. That (32) is an approximate equilibrium condition with respect to the originally given data follows from (33). It is noted, however, that the "true" optimum $({x^*}_1, {x^*}_2)$ may not satisfy $\xi_{0i} \leqq x^* \leqq {\xi_i}^0$.

[26] For proof see the author's discussion paper, *op. cit.*, Appendix VIII.

(33) $|f_1(x_1) + f_2(x_2) - y^*| < max[n, 4] + 2m = d,$

for all (x_1, x_2) such that $\xi_{0i}(\pi^*) \leqq x_i \leqq \xi_i{}^0(\pi^*)$ $(i = 1,2)$ and $x_1 + x_2 = x^0$. This implies that the allocation error can be made arbitrarily small by choosing sufficiently large m and n, since $d/y^0 \to 0$ as $m \to \infty$ and $n \to \infty$. (Note that $d/y^0 = 3/m$ if $m = n \geqq 4$.)

Let us next state the price adjustment scheme in more detail. To do this, we need to specify (i) a rule for choosing the initial price and (ii) a rule for choosing a new price at each step of adjustments. (The law of supply and demand indicates the direction of the adjustment, but not its magnitude.) We assume that rules are chosen in such a way that the largest possible number of adjustment steps be minimized. In other words, we assume that the initial price is equal (or close) to the midpoint of the segment $P(= n/2)$ and that the subsequent choice of new prices follows the *bisectioning rule* approximately. If $n = 2^k$ for some positive integer k, then it is possible to follow the bisectioning rule precisely:

$\pi(1) = 2^{k-1},$

(34) $\pi(t) = \pi(t-1) \pm 2^{k-t},$

where $\pi(1)$ is the initial price and $\pi(t)$ is the price chosen at the t^{th} step of adjustments. It is clear that for this special case the largest possible number of steps is equal to $k = log_2 n$. If n is not a power of 2, the bisectioning rule may be followed approximately, and the largest possible number of steps is approximately equal to $log_2 n$.

The actual number of steps needed to reach equilibrium depends on the exogenously given data (g_1 and g_2). It is even possible, if not likely, that the initial price hits the equilibrium; no adjustment is necessary then. A sophisticated treatment would compute the expected number of steps needed, assuming that some probability distribution on the set G is given. In this paper, however, we estimate only the upper bound of the number of steps.

2. The amount of information sent and received by each firm in one step is $log\, n$ (for transmitting price π) plus $2 log\, m$ (for transmitting the demand correspondence$(\xi_{0i}, \xi_i{}^0)$).

Hence, we may assert that

(35) $C_2 \leqq 2 log_2\, n \;\; (log\, n + 2\, log\, m),$

where C_2 is the total cost of communication in this system. If, in particular, $m = n$, then we have

(36) $C_2 \leqq 6(log\, m)^2 / \, log\, 2.$

C. Comparison of the Two Systems

Table I lists the lower and the upper bounds of C_1 and the upper bound of C_2 for some values of m (we assume $m = n$ throughout this section). If m is set greater than 100, that is, if it is required that the relative allocation errors be less than 3 percent ($= 3/100$; see (33)), then the cost of communication in the price mechanism is definitely lower than that in the centralized system. However, if we allow somewhat higher relative allocation errors, then it is possible that the centralized decision making is cheaper with respect to the cost of communication than the price mechanism.

IV. CONCLUSIONS AND SOME FURTHER REMARKS

In the preceding sections we considered a simple problem of allocating resources between two firms. We introduced a method for approximating economic data with finite and discrete sets and computed the cost of internal communication in two economic systems seeking efficient allocations. The comparison of the cost of communication confirms that the decentralized price mechanism is more economical if the required accuracy of allocation is high, but it also suggests that the centralization of information may pay if the required accuracy is relatively low.

We are interested in comparing different economic systems, since in the real world the environment is often non-classical (e.g., externalities like pollution) and the market mechanism is likely to fail in such an environment. Further, we are interested in comparing different systems with respect to the cost of communication, or more generally with respect to transaction or organizational costs, since these costs affect the systems' overall efficiency (if one takes the normative viewpoint) or their viability (if one takes the descriptive viewpoint).

The main purpose of this paper, however, is not to derive a general conclusion about the preference on the centralized system and the price mechanism, nor to consider allocation problems in the presence of externalities, but to propose certain analytical tools that might turn out to be useful for further studying these problems. Partly for this reason and partly for the reason of mathematical difficulty, we kept our model within a very simple and limited framework. Below we briefly consider some of the possible extensions of our model: (i) The "command system," an adjustment mechanism that is like the price mechanism except that the roles of price and quantity are interchanged: Marglin once pointed out that the price mechanism and the command system are informationally equivalent.[27] I confirmed this (with respect to the cost of communication) by using a method similar to the one used here. (ii) The case of many firms: All the conclusions obtained in this paper will be carried through, since the cost of communication is linear in the number of firms. However, if more than two firms are present, the possibility of systems other than those considered here arises (e.g., partially centralized economic systems). (iii) The case of many input (and output) commodities: It seems mathematically difficult to evaluate $|G|$, and also it is difficult to extend the "bisection rule" to this case. (iv) The case of increasing returns or non-concave production functions: It is possible to evaluate $|G|$ for the single-input---single-output case, so that the centralized system can be handled easily. The difficulty lies in devising a decentralized adjustment process in a

[27] S. A. Marglin, "Information in Price and Command Systems of Planning," in J. Margolis and H. Guitton, eds., *Public Economics* (New York: Macmillan, 1969), Ch. 3.

suitable way. (v) Sequential communication, computation, and decision making, with the possibility of variable degrees of accuracy: This is an interesting but seemingly difficult case, on which we have made no investigation.

The central method in this paper is discrete approximation of economic data. It was introduced into our model based on the observation that any data used for information processing are fuzzy and contain errors and that decision making based on such data must be fuzzy too. The need for considering explicitly the accuracy of data was demonstrated by the sensitivity to it of the relative cost of communication in the two systems. It is noted, however, that our method --- discrete approximation --- is not the only way to express inaccuracy. For example, it is possible to express it by adding to economic data error terms distributed probabilistically. Our method is useful for the present study, but it also brings inconveniences. First of all, if information is processed exclusively by digital mechanical devices, then discretization of data is natural, since this is the form in which a digital device handles information. But in the real world, of course, information is not always processed in digital forms. Apart from this point the usefulness of discrete approximation lies in that (i) it makes everything finite and elementary so that Shannon's measure can easily be used and (ii) that it makes the adjustment rule for the price mechanism very simple. One of its shortcomings may be that discrete approximation leaves irregularly distributed residuals, which cannot be dealt with in a simple setting. In particular, as seen in Section II, it is difficult (if not impossible) to approximate concave functions by using discrete data. Whether there is any method for expressing the fuzziness of data that is better than discrete approximation seems to be an open question.

APPENDIX A: COMMUNICATION AND THE AMOUNT OF INFORMATION

This appendix summarizes the part of communication theory that is needed for this work.

For the present purpose communication is to transmit a message chosen from the set of possible messages.[28] We assume that the set is known both to the sender and the receiver of messages and that the set is finite. Denote the set by

(A. l) $\mathrm{A} = \{a_i\} = \{a_1, a_2, \dots, a_n\}$,

where a_i is a possible message.[29] We assume that the probability for message a_i to occur is p_i:

(A.2) $P = \{p_i\} = \{p_1, p_2, \dots, p_n\}, p_i \geqq 0, \sum p_i = 1.$

A message is transmitted through a *channel* in a coded form. The means of *coding* could be letters, numerals, words, sounds, voices, electric currents or pulses, etc. The power of

[28] For example, consider today's change in a stock price. The set of possible messages may be composed of three elements: UP, DOWN, UNCHANGED.

[29] In the above example, we might set $a_1 = \mathrm{UP}$, $a_2 = DOWN$, $a_3 = UNCHANGED$, and $n = 3$.

communication theory lies in its independence of the choice of a particular channel and of the choice of particular coding. Consider *binary coding*; a coded word (or *code)* that represents a message is a sequence of binary digits (*bits*), 0 and 1. The *length* of a code is the number of bits composing it.[30]

The cost of transmitting message a_i depends on the length of its code. The length of each code, of course, is determined by how each message is coded. It is advantageous to give short codes to messages transmitted frequently and to give long codes to messages transmitted less frequently, since this minimizes the *average length* of a code.

It has been shown by Shannon that, if each of the successive messages occurs independently, then the greatest lower bound of the average length of a code is equal to[31]

(A.3) $H(P) = -\sum p_i \, log \, p_i \; (\geqq 0)$.

The right side of (A.3) denotes *the amount of information* transmitted by a message. It is also called the entropy of the system $\{A, P\}$, the measure of the "degree of uncertainty" of the system. The greater is the degree of uncertainty, the more is the amount of information needed to resolve it. If, in particular, each message occurs with equal probability so that $p_i = 1/n$ for all i then

(A.4) $H(P) = H(n) = log \, n$,

as can easily be derived from (A.3).[32] It is noted that (A.4) is the maximum of (A.3) in P. In other words, the amount of information given by a message is maximized when one knows nothing about which message is likely to occur.

The *capacity* of a channel is the number of bits that it can transmit per unit of time. Therefore, if the capacity is denoted by C, then the greatest lower bound of time needed to transmit information in the amount $H(P)$ is equal to $H(P)/C$. A remarkable result obtained by Shannon is that the above relation can be extended to noisy channels.[33] A noisy channel transmits bits with possible errors: 0 as 1 or 1 as 0. If the probability law governing errors in transmission is known, then it is possible to define the capacity from that probability law *only*, so that the relation among the capacity, the amount of information, and the transmission time is precisely the same as in noiseless channels.

The cost of communication includes the cost of constructing and operating a channel that

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30 In the above example, we might code: $\mathrm{UP} = 00$, $\mathrm{DOWN} = 01$, and $\mathrm{UNCHANGED} = 1$. The length of the first two codes is 2, while that of the last is 1.

31 Shannon, *op. cit.*, Theorem 9; or R. Ash, *Information Theory* (New York: John Wiley, 1965), Theorems 2.5.1 and 2.5.2. The choice of the base of the logarithm determines the unit of measuring the amount of information. In binary coding the most convenient base is 2; the amount is then expressed in terms of the number of bits.

32 The average length of the codes in note 30 is $(2 + 2 + 1)/3 = 1.667$ if each of the three messages occurs with equal probability. Since the greatest lower bound for the average length is $H(3) = log_2 3 = 1.585$, it is possible to shorten the average length by improving coding. This is done by coding a sequence of messages at one time rather than by coding individual messages separately.

Equation (A.4) can easily be verified if n is a power of 2. The simplest case is $n = 2$, i.e., $A = \{a_1, a_2\}$. Then, say we code, $a_1 = 0$, and $a_2 = 1$. Since the length of each code is 1, so is the average length; this agrees with (A.4): $log_2 2 = 1$. If $n = 4$, then the coding may be $a_1 = 00$, $a_2 = 01$, $a_3 = 10$, $a_4 = 11$; the average length is $2 \; (= log_2 4)$.

33 Shannon, *op. cit.*, Theorem 11, or Ash, *op. cit.*, Theorem 3.5.1.

transmits messages. In considering the cost, we may take two viewpoints, short-run and long-run. From the short-run viewpoint we suppose that a channel, noisy or noiseless, of some capacity has already been given. Then the amount of information determines the transmission time, which together with the operating cost of the channel constitutes the (variable) cost of communication. From the long-run viewpoint we suppose that a channel is yet to be designed and constructed. The choice of an optimum channel depends upon various engineering and economic factors. However, since the amount of information is defined independently of channels, we can consider it as the determinant of the long-run cost of communication. Therefore, the cost of communication may be considered independently of channels in spite of the fact that the cost is attached to channels.[34]

APPENDIX B: PROOFS

B.1: Proof of (12) *and* (13). Since f belongs to F, (8) implies that $0 \leqq f(j) - f(j-1) \leqq p^0$. We write (9) as

(b1) $g(j-1) - 1 < f(j-1) \leqq g(j-1),$

$g(j) - 1 < f(j) \leqq g(j).$

From the above inequalities we derive

(b2) $0 \leqq f(j) - f(j-1) < g(j) - g(j-1) + 1,$

$g(j) - 1 - g(j-1) < f(j) - f(j-1) \leqq p^0 = n.$

Since $g(j)$'s are integers, we obtain $0 \leqq g(j) - g(j-1) \leqq n$, which is (12) (See (10).) Also, (13) follows from (b.2) and (10).

B.2: Proof of (*21*) *and* (*22*). *By the definition of* A_0 *and* A^0 (see (18)),

(b2) $A_0(\pi - 1) \supset A_0(\pi), \qquad A^0(\pi - 1) \subset A^0(\pi).$

Then (19) implies the first and the third inequalities of (21). To show the second inequality of (21), suppose that it did not hold:

$\xi^0(\pi) < \xi_0(\pi - 1)$. Then there would exist a $j \in M$ such that

$\xi^0(\pi) < j \leqq \xi_0(\pi - 1)$. Since $\xi_0 \in A_0$ and $\xi^0 \in A^0$, we would have $h(j) < \pi$ and $h(j) > \pi - 1$, which is a contradiction since $h(j)$ is an integer.

B.3:Proof of (20). Let $\xi_0 = \xi_0(\pi)$ and $\xi^0 = \xi^0(\pi)$. The first and the third propositions of (20) follow directly from $\xi_0 \in A_0(\pi)$ and $\xi^0 \in A^0(\pi)$ respectively. It remains to show the

[34] The relation of the organization problem to communication is in a sense analogous to that of the location problem to transportation. (Both of these are "network" problems.) In the location problem the cost of transportation is expressed in terms of, e.g., passenger-miles or ton-miles without specifying a particular mode of transportation, and this much specification is usually sufficient to solve the location problem. It is noted that the concept of "accuracy" is of little practical importance in transportation (however, one could think of transmission of electrical energy on wires with some loss), while in communication the degree of accuracy is a basic notion. The usefulness of communication theory to the organization problem lies in that, despite the presence of inaccuracy in communication, it makes it possible to handle the cost of communication in the organization problem in the same way as one handles the cost of transportation in the location problem. (There is another outcome from comparing communication to transportation: J. Marschak, "Economics of Inquiring, Communicating, Deciding," p. 9.)

second proposition of (20). To do this, observe first that the proposition presupposes $\xi_0 < \xi^0$. We then have $\xi_0 < m$ and $\xi^0 < 0^0$. It follows from (18) and (19) that

(b4) $\quad h(\xi_0 + 1) \leqq \pi, \qquad h(\xi^0) \geqq \pi.$

(i) Assume that $\xi_0 + 1 \leqq j$. Then from (13) and the concavity of f we get

$$h(j) - 1 < f(j) - f(j-1) \leqq f(\xi_0 + 1) - f(\xi_0) < h(\xi_0 + 1) + 1.$$

In view of (b4) we have $h(j) \leqq h(\xi_0 + 1) + 1 \leqq \pi + 1$. (ii) Assume that $j < \xi^0$. Then again from (13) and the concavity of f we get

$$h(\xi^0) - 1 < f(\xi^0) - f(\xi^0 - 1) \leqq f(j) - f(j-1) < h(j) + 1.$$

Hence, we have $\pi - 1 \leqq h(\xi^0) - 1 \leqq h(j)$. Combining (i) and (ii) above, we obtain the second proposition of (20). (It is noted that, if $\xi_0 + 1 = \xi^0$, then $h(\xi_0 + 1) = h(\xi^0) = \pi$ from (b4).)

B.4: Proof of (25). (i) Let $g \in G_*$. Define an extension f of g to X by $f(x) =$(the right side of (16)). Then clearly $g = g_f$ and $f \in F$; i.e., $g \in G = G_F$. This proves the first half of (25). (ii) Let $g \in G = G_F$. Then there exists an $f \in F$ such that $g = g_f$. Inequality (13) together with the concavity of f implies that $h(j) - 1 < f(j) - f(j-1) \leqq f(j-1) - f(\mathrm{j} - 2) < h(j-1) + 1, j = 2, \dots, m$, where $h = h_g = h_f$. From this we get $h(j-1) + 2 > h(j)$ (since $h(j)$'s are integers). This implies $g \in G^*$, proving the second half of (25).

B.5: Proof of (26). (i) For a given $g \in G_*$ and the corresponding $h = h_g$, define a function h' by

(b5) $\quad h'(j) = h(j) - (j-1), \qquad j \in M.$

Then h' is strictly decreasing (since h is non-increasing) and maps M into a set $N' = N \cup \{0, -1, -2, \dots, -(m-1)\}$. Conversely, if h' is a strictly decreasing function that maps M into N', then the function h obtained from h' via (b5) is non-increasing and maps M into N. Hence, the function g obtained from this h via (11) belongs to G_*. In other words, correspondence between g and h' determined by (10) and (b5) is one-one. Therefore, $|G_*|$ is equal to the number of strictly decreasing functions h' that map M into N'. Such an h' is specified by first choosing m elements, say, (k_j) from N' without repetition, rearranging them into, say, (k'_j), which is in the decreasing order, and letting $h'(j) = k'_j$. Therefore, $|G_*| =$(the number of combinations of m objects taken from the set of $m + n$ distinct objects), since $|N'| = m + n$. This proves the first equation of (26). (ii) The second equation of (26) may be proved similarly if one replaces (b5) by $h(j') = h(j) - 2(j-1), \qquad j \in M$.

B.6: Proof of (29). We prove the second equation of (29). (The first equation may be proved analogously.) Put $m = n$. Then, noting that $3m - 1/2 = (m + 1/2) + (2m - 1/2) + (-1/2)$, we have (the first term of the right side of (33))$= log\{(3m-1)^{3m-1/2}/$

$m^{m+1/2}(2m-1)^{2m-1/2}\} = log\{(3-1/m)^{3m}(2-1/m)^{-(2m-1/2)}(3m-1)^{-1/2)}\} =$

$mU(m)$, where $U(m) = log\{(3-1/m)^3(2-1/m)^{-(2-1/2m)}\} - 1/2\, m \cdot log(3m-1)$.

Since $U(m) \rightarrow \log(3^3/2^2) = \log 27/4$ as $m \rightarrow \infty$, we have derived the second equation of (29).