REGULARITY CONDITIONS FOR CONCAVE PROGRAMMING IN FINITE DIMENSIONAL SPACES

Hajime Oniki

Technical Report No. 6

Prepared under Contract No. N00014-67-A-0298-0019
Project No. NR 047-004
For Office of Naval Research

This document has been approved for public release and sale; its distribution is unlimited.

Reproduction in whole or in part is permitted for any purpose of the United States Government.

Harvard University,
1737 Cambridge Street, Room 401
Cambridge, Massachusetts 02138

April 1971

REGULARITY CONDITIONS FOR CONCAVE PROGRAMMING IN FINITE DIMENSIONAL SPACES*/

Hajime Oniki

I. Introduction

It is well known that problems of constrained maximum can be reduced to saddle-point or quasi-saddle-point problems by the method of Lagrangian multipliers. This method is a useful tool for economic analysis, since it reveals in a natural way the notion of shadow prices accompanying the constraint functions of the maximum problems. Further, it is useful for computational purposes.

It is also known, however, that the reduction may not be possible unless some regularity conditions are satisfied by the functions involved in constrained maximum problems. Kuhn and Tucker [4] first considered nonlinear maximization problems with inequality constraints and proved that the reduction is possible whenever the functions involved are differentiable and the constraint functions satisfy a regularity condition called constraint qualification. Later, Arrow, Hurwicz, and Uzawa [1] examined various constraint qualifications and showed a condition which is the weakest of this kind in the following sense: the condition is necessary and sufficient for a differentiable constrained maximum problem to be reducible for any differentiable objective function.[1/] They also showed

a number of simpler constraint qualifications sufficient for the reduction.

On the other hand, Uzawa [6] treated the case where the functions involved are concave but not necessarily differentiable and proved that a constraint qualification originally given by Slater and an extension of it are sufficient for the reduction to be valid.[2/] It is an interesting question whether or not the weakest constraint qualification can be found for concave problems as done in [1] for differentiable problems.

It is noted that constraint qualifications are stated as conditions on the constraint functions only; they do not involve the objective function. It may therefore be expected that even though the constraint functions do not satisfy the weakest constraint qualification, the reduction might be possible for some, though not for all, objective functions. In fact, this is true even if the objective and the constraint functions are restricted to be differentiable and concave.[3/] It is therefore interesting to ask what properties of the objective and constraint functions determine the reducibility of constrained maximum problems. Recently, Rockafellar [5] gave an answer to this question in an indirect way; he showed by using a parametric approach that, roughly speaking, the case where the reduction is impossible occurs when the Lagrangian multipliers accompanying the constraint functions become infinite.[4/]

The purpose of the present paper is to study the two questions stated above for constrained maximum problems with concave but not necessarily differentiable functions. The discussion will be restricted to the case where the number of variables entering the objective and the constraint functions is finite and the number of constraint functions is also finite. It will be shown in Section IV that a condition on the objective and the constraint functions (Regularity Condition (46)) is necessary and sufficient for the reduction (Theorem 2). This solves the second question in a sense completely. On the other hand, the first question is answered partially in this paper. In Section V a considerably weak constraint qualification (Constraint Qualification (69)) is presented, from which Slater-Uzawa's conditions and another condition ensuring the reduction can be derived (Theorem 3).

Most results obtained in this paper depend on the fact that a concave function possesses all the (one-sided) directional derivatives of any order at any (interior) point of its definition. The regularity conditions below are stated in terms of directional derivatives.[5/] In fact, it turns out that we need to use directional derivatives not only of the first order but also of higher orders. It may be of some use to summarize the properties of directional derivatives to the extent needed for the present work.[6/] Sections II and III are devoted mainly for this purpose.

II. Preliminaries: The Directional Derivatives of a Concave Function

1. Let $\varphi(x)$ be a real-valued concave function defined for any real n-vector $x = (x_1,\ldots,x_n)$; i.e.,

(1) $$\varphi(tx + (1-t)x') \geqq t\varphi(x) + (1-t)\varphi(x')$$

for all x, x', and $0 \leqq t \leqq 1$.

Then it can be shown that $\varphi(x)$ is continuous for all x and that

(2) $\dfrac{\varphi(x + \mu\xi) - \varphi(x)}{\mu}$ is decreasing in $\mu > 0$ for all x and ξ.[7/]

From (2) we can assert that the expression

(3) $$d\varphi(x;\xi) = \sup_{\mu>0} \frac{\varphi(x + \mu\xi) - \varphi(x)}{\mu}$$

$$= \lim_{\mu\downarrow 0} \frac{\varphi(x + \mu\xi) - \varphi(x)}{\mu}$$

exists for all x and ξ. We call (3) the directional derivative of φ at x with increment ξ. Thus, a concave function defined for all x possesses a unique directional derivative at any x with any increment ξ.

From the definition of $d\varphi(x,\xi)$, it is immediately seen that for any $\mu > 0$,

(4) $$d\varphi(x;\xi) \geqq \frac{\varphi(x + \mu\xi) - \varphi(x)}{\mu}$$

In particular (put $\mu = 1$),

(5) $\varphi(x + \xi) \leqq \varphi(x) + d\varphi(x;\xi)$, for all x and ξ.

The definition of $d\varphi(x;\xi)$ indicates that in general $d\varphi(x;\xi) \leqq +\infty$. For the present case where $\varphi(x)$ is defined for all x,[8] it can be shown that

$$d\varphi(x;\xi) < +\infty, \text{ for all } x, \xi.[9]$$

The following relation is then an immediate consequence of (3) and (4):

(6) $\varphi(x+\mu\xi) = \varphi(x) + d\varphi(x;\xi)\mu + o(\mu)$, where $o(\mu) = o(\mu;x,\xi)$, $o(\mu) \leqq 0$, and $o(\mu)/\mu \uparrow 0$ as $\mu \downarrow 0$.

The directional derivative is positively homogeneous and sub-additive:

(7) $d\varphi(x;\lambda\xi) = d\varphi(x;\xi)\lambda$, for all $\lambda \geqq 0$,

(8) $d\varphi(x;\xi+\eta) \geqq d\varphi(x;\xi) + d\varphi(x;\eta)$, for all ξ,η. Therefore, $d\varphi(x;\xi)$ is concave in ξ.[10]

In the remaining part of the paper, we shall often write for simplicity as

(9) $d\varphi(x;\xi) = d\varphi(\xi)$, etc., if x is held fixed so that no confusion is likely. Also, directional derivatives will sometimes be called derivatives.

2. From the fact that the directional derivative $d\varphi(\xi) = d\varphi(x;\xi)$ itself is concave in ξ, it follows that higher order derivatives always exist; e.g.,

$$d^2\varphi(x,\xi^1;\xi) = d(d\varphi)(x,\xi^1;\xi)$$

$$= \sup_{\mu>0} \frac{d\varphi(x;\xi^1 + \mu\xi) - d\varphi(x;\xi^1)}{\mu}$$

$$d^3\varphi(x,\xi^1,\xi^2;\xi) = d(d^2\varphi)(x,\xi^1,\xi^2;\xi)$$

$$= \sup_{\mu>0} \frac{d^2\varphi(x,\xi^1;\xi^2 + \mu\xi) - d^2\varphi(x,\xi^1;\xi^2)}{\mu},$$

etc. In general,

$$d^i\varphi(x,\xi^1,\ldots,\xi^{i-1};\xi) = d(d^{i-1}\varphi)(x,\xi^1,\ldots,\xi^{i-1};\xi)$$

(10)

$$= \sup_{\mu>0} \frac{d^{i-1}\varphi(x,\xi^1,\ldots;\xi^{i-1} + \mu\xi) - d^{i-1}\varphi(x,\xi^1,\ldots;\xi^{i-1})}{\mu}$$

Namely, the i-th order derivative $d^i\varphi(x,\xi^1,\ldots,\xi^{i-1};\xi)$ is the derivative of $d^{i-1}\varphi(x,\xi^1,\ldots,\xi^{i-2};\xi^{i-1})$ at ξ^{i-1} with increment ξ.

In general, the argument of the i-th order derivative is composed of (i+1) vectors $x,\xi^1,\ldots,\xi^{i-1},\xi$. To simplify the notation, we introduce

(11) $$\xi^{(i)} = (\xi^1,\ldots\xi^i),$$

and write

$$d^i\varphi(x,\xi^1,\ldots\xi^{i-1};\xi) = d^i\varphi(x,\xi^{(i-1)};\xi), \text{ etc.}$$

Also, as done in (9), we shall sometimes write as

$$d^i\varphi(x,\xi^{(i-1)};\ \xi) = d^i\varphi(\xi^{(i-1)};\xi) = d^i\varphi(\xi), \text{ etc.},$$

if the omitted arguments are held fixed so that no confusion is likely.

If $\varphi(x)$ is differentiable at x (in the ordinary sense), then (3) implies

(12) $\quad d\varphi(x,\xi) = \varphi_x(x)\cdot\xi,$

where $\varphi_x(x)$ denotes the n-vector of the partial derivatives of $\varphi(x)$ with respect to x_i (i=1,...,n). In particular, if $\varphi(x)$ is linear, say $\varphi(x) = ax$, where a is an n-vector, then

(13) $\quad d\varphi(x,\xi) = a\xi.$

Therefore, if $\varphi(x)$ is differentiable, we have

(14) $\quad d^i\varphi(x,\xi^{(i-1)};\xi) = \varphi_x(x)\cdot\xi = d\varphi(x;\xi)$, for all i. This points out the fact that if the function in question is differentiable as well as concave, then it is unnecessary to consider its higher order directional derivatives.

LEMMA 1:

(15) $\quad d^2\varphi(\xi;\alpha\xi) = \alpha d\varphi(\xi)$, for any vector ξ and any real number α, and

(16) $\quad d^2\varphi(\xi^1;\xi^2) \geq d\varphi(\xi^2)$, for any vectors ξ^1 and ξ^2.

Proof:

$$d^2\varphi(\xi;\alpha\xi) = \sup_{\mu>0} \frac{d\varphi(\xi + \mu\alpha\xi) - d\varphi(\xi)}{\mu}$$

$$= \sup_{|\frac{1}{\alpha}|>\mu>0} \frac{(1 + \mu\alpha)d\varphi(\xi) - d\varphi(\xi)}{\mu} \qquad \text{(by (7))}$$

$$= \alpha d\varphi(\xi), \text{ and}$$

$$d^2\varphi(\xi^1;\xi^2) = \sup_{\mu>0} \frac{d\varphi(\xi^1 + \mu\xi^2) - d\varphi(\xi^1)}{\mu}$$

$$\geq \sup_{\mu>0} \frac{d\varphi(\xi^1) + \mu d\varphi(\xi^2) - d\varphi(\xi^1)}{\mu} \quad \text{(by (8),(7))}$$

$$= d\varphi(\xi^2). \quad \text{(QED)}$$

The following is a generalization of (6) and (5):

LEMMA 2:

(a) $$\varphi(x + \sum_{i=1}^{k} (\mu^1 \ldots \mu^i)\xi^i) = \varphi(x) + \sum_{i=1}^{k} d^i\varphi(x, \xi^{(i-1)}; \xi^i)(\mu^1 \ldots \mu^i) + \sum_{i=1}^{k} o^i(\mu^i)(\mu^1 \ldots \mu^{i-1}),$$

where $\xi^{(i-1)} = (\xi^1, \ldots, \xi^{i-1})$, $o^i(\mu^i) \leq 0$, and $o^i(\mu^i)/\mu^i \uparrow 0$ as $\mu^i \downarrow 0$, for $i=1,\ldots,k$ $(k > 0)$.[11/]

(b) Furthermore, for any $\varepsilon > 0$, there exist $\mu^i > 0$ $(i=1,\ldots,k)$ such that

(18) $$-\sum_{i=1}^{k} o^i(\mu^i)(\mu^1 \ldots \mu^{i-1}) < \varepsilon(\mu^1 \ldots \mu^k).$$

(For a proof, see the appendix.)

LEMMA 3: Let $\varphi(x) \geq 0$. If for $i=1,\ldots,k$,

$$\varphi(x + \sum_{r=1}^{i} (\mu^{1i} \ldots \mu^{ri})\xi^r) \geq 0, \text{ for } \underline{\text{some}}\ \mu^{ri} > 0 \ (r=1,\ldots,i),$$

then, for $i=1,\ldots,k$,

(19) $$\varphi(x + \sum_{r=1}^{i} (\lambda^1 \ldots \lambda^r)\xi^r) \geq 0, \text{ for } \underline{\text{any}}\ \lambda^r \text{ such that}$$

$$0 < \lambda^r < \mu^r \equiv \min_{(r \leq i \leq k)} \mu^{ri} \quad (r=1,\ldots,k).$$

(For a proof, see the appendix.)

LEMMA 4: Let $\varphi(x)$ be a concave function defined for all x. Then, $\varphi(\bar{x}) = \max_x \varphi(x)$ if and only if $d\varphi(\bar{x};\xi) \leq 0$ for all ξ.

Proof: Assume $\varphi(\bar{x}) = \max_x \varphi(x)$. Choose an arbitrary ξ and $\mu > 0$, and let $x = \bar{x} + \mu\xi$. Then by (6),

$$\varphi(\bar{x}) \geq \varphi(x) = \varphi(\bar{x} + \mu\xi) = \varphi(\bar{x}) + d\varphi(\bar{x};\xi)\mu + o(\mu).$$

I.e.,

$$d\varphi(\bar{x};\xi) \leq -o(\mu)/\mu.$$

Since $o(\mu)/\mu \uparrow 0$ as $\mu \downarrow 0$, we get $d\varphi(\bar{x};\xi) \leq 0$.

On the other hand, assume $d\varphi(\bar{x};\xi) \leq 0$ for all ξ. Choose an arbitrary x, and let $x = \bar{x} + \xi$. Then (5) yields

$$\varphi(x) = \varphi(\bar{x} + \xi) \leq \varphi(\bar{x}) + d\varphi(\bar{x};\xi) \leq \varphi(\bar{x}). \quad \text{(QED)}$$

III. Constrained Maximum Problems

1. Let $f(x)$ be a real-valued concave function defined for all n-vector $x=(x_1,\ldots,x_n)$ and $g(x) = (g_1(x),\ldots,g_m(x))$ be an m-vector of real-valued concave functions $g_j(x)$ $(j=1,\ldots,m)$ defined for all x. (These assumptions on $f(x)$ and $g(x)$ will be maintained throughout the rest of the paper, unless stated otherwise.) In what follows, we shall deal with problems of finding a vector $\bar{x}$ which maximizes the objective function $f(x)$ subject to the constraints $g(x) \geq 0$.[12/] (Such $\bar{x}$ will be called an optimum.)

The following theorem states a necessary and sufficient condition for a vector $\bar{x}$ to be an optimum (whether the Lagrangian

reduction, to be defined later, is possible or not). Define

(20) $\quad K = \text{the closure of } \left\{\xi \,\middle|\, g(\bar{x} + \mu\xi) \geq 0, \text{ for some } \mu > 0\right\}.$

Then,

THEOREM 1: $\bar{x}$ maximizes $f(x)$ subject to $g(x) \geq 0$ if and only if $g(\bar{x}) \geq 0$ and $\xi \epsilon K$ implies $df(\bar{x},\xi) \leq 0$.

Proof: (i) ('only-if' part) Assume that $\bar{x}$ maximizes $f(x)$ subject to $g(x) \geq 0$. Let ξ satisfy $g(\bar{x} + \mu'\xi) \geq 0$ for some $\mu' > 0$. From the concavity of $g(x)$ and the assumption $g(\bar{x}) \geq 0$,

$$g(\bar{x} + \mu\xi) \geq 0, \qquad \text{for all } 0 < \mu < \mu'.$$

Then from (6) and the optimality of $\bar{x}$, we get

$$f(\bar{x}) \geq f(\bar{x} + \mu\xi) = f(\bar{x}) + df(\bar{x};\xi)\mu + o(\mu), \text{ or}$$

$$df(\bar{x};\xi) \leq -o(\mu)/\mu, \text{ for all } 0 < \mu < \mu'.$$

Passing to the limit, we obtain $df(\bar{x},\xi) \leq 0$. Since $df(\bar{x},\xi)$ is continuous in ξ, we conclude that $\xi \epsilon K$ implies $df(\bar{x},\xi) \leq 0$.

(ij) ('if' part) Assume that $g(\bar{x}) \geq 0$ and that $\xi \epsilon K$ implies $df(\bar{x},\xi) \leq 0$. Let x satisfy $g(x) \geq 0$ and put $\xi = x - \bar{x}$. Then (20) implies $\xi \epsilon K$, and hence by the assumption one has $df(\bar{x},\xi) \leq 0$. But from (5) one concludes that

$$f(x) = f(\bar{x} + \xi) \leq f(\bar{x}) + df(\bar{x},\xi) \leq f(\bar{x}). \qquad \text{(QED)}$$

2. In the sequel, we shall seek conditions which ensure the reduction of constrained maximum problems to saddle-point problems:

LAGRANGIAN REDUCTION: Let $\bar{x}$ be an optimum. Then, the functions $f(x)$ and $g(x)$ are <u>Lagrange reducible</u> at $\bar{x}$ if there exists an

m-vector $\bar{y} \geq 0$ (Lagrangian multipliers) such that $(\bar{x},\bar{y})$ is a saddle-point of the Lagrangian form

(21) $\quad L(x,y) = f(x) + yg(x)$

subject to $y \geq 0$. I.e., $(\bar{x},\bar{y})$ satisfies

(22) $\quad \bar{y} \geq 0,$

(23) $\quad L(x,\bar{y}) \leq L(\bar{x},\bar{y}) \leq L(\bar{x},y)$, for all x and all $y \geq 0$.

The fact that the Lagrangian reduction is not always possible (even for the case where the objective and the constraint functions are concave and differentiable) is seen from the following example: Let n=m=1, and

(24) $\quad f(x) = x,$

$\quad g(x) = -x^2$; then

$\quad L(x,y) = x + yx^2$

Then, f(x) and g(x) are both concave, $\bar{x} = 0$ is an optimum, and yet the first inequality of (23) is not satisfied for any $\bar{y} \geq 0$, since $L(x,\bar{y}) = x(1 - \bar{y}x) > 0 = L(\bar{x},\bar{y})$ for $0 < x < 1/\bar{y}$ (if $\bar{y} > 0$) or for $x > 0$ (if $\bar{y} = 0$).

3. For the present case where the functions f(x) and g(x) are assumed concave, the saddle-point property (22) and (23) above can be expressed in a different but equivalent way in terms of the directional derivatives of f(x) and g(x).

To do this, let $\bar{x}$ be an optimum. We divide the components of $g(\bar{x}) \geq 0$ into two groups (by reordering the components if necessary)

in such a way that

$$(25) \qquad g(\bar{x}) = (g_1(\bar{x}),\ldots,g_m(\bar{x})) = (g_E(\bar{x}),\ g_F(\bar{x})),$$

where $E = \{j \mid g_j(x) = 0,\ \ 1 \leqq j \leqq m\}$,

$F = \{j \mid g_j(x) > 0,\ \ 1 \leqq j \leqq m\}$.

Namely, $g_E(\bar{x})$ is composed of the effective constraints and $g_F(\bar{x})$ of the ineffective constraints. The directional derivatives of the components of $g(\bar{x})$ will be written as vectors in a similar way:

$$(26) \qquad dg(\bar{x},\xi) = (dg_1(\bar{x},\xi),\ldots,\ dg_m(\bar{x},\xi)) = (dg_E(\bar{x},\xi),\ dg_F(\bar{x},\xi)),\ \text{etc.}$$

LEMMA 5: Let $L(x,y)$ be defined by (21). Then, $(\bar{x},\bar{y})$ is a saddle-point of $L(x,y)$ if and only if there exists $\bar{y}_E \geqq 0$ such that

$$(27) \qquad df(\bar{x},\xi) + y_E dg_E(\bar{x},\xi) \leqq 0, \text{ for all } \xi,$$

where $\bar{x}$ satisfies $g(\bar{x}) \geqq 0$ and (25).

Proof: First, assume that $(\bar{x},\bar{y})$ satisfies (22) and (23). Then from the second inequality of (23), $\bar{y}g(\bar{x}) \leqq yg(\bar{x})$ for all $y \geqq 0$. Hence, $g(\bar{x}) \geqq 0$. Also, by putting $y = 0$, one obtains $\bar{y}g(\bar{x}) \leqq 0$. Since $\bar{y} \geqq 0$, one gets $\bar{y}g(\bar{x}) = 0$. In view of $g(\bar{x}) \geqq 0$, we may write down $g(\bar{x})$ as in (25). Further, write $\bar{y} = (\bar{y}_E,\ \bar{y}_F)$ in the corresponding way. From $g_F(\bar{x}) > 0$, it follows that $\bar{y}_F = 0$. Now the first inequality of (23) can be written as

$$(28) \qquad f(x) + \bar{y}_E g_E(x) \leqq f(\bar{x}) + \bar{y}_E g_E(\bar{x}), \quad \text{for all } x.$$

In view of LEMMA 4, one obtains (27).

On the other hand, assume that $\bar{x}$ satisfies $g(\bar{x}) \geqq 0$ and (25), and that there exists $\bar{y}_E \geqq 0$ for which (27) holds. LEMMA 4 implies (28). Define $\bar{y} = (\bar{y}_E, 0) \geqq 0$. Then the first inequality of (23) holds. Also, one has $\bar{y}g(\bar{x}) = 0$. Since $g(\bar{x}) \geqq 0$, it follows that $0 = \bar{y}g(\bar{x}) \leqq yg(\bar{x})$ for all $y \geqq 0$, which implies the second inequality of (23). (QED)

It is expected from this LEMMA that the Lagrangian reducibility depends upon the property of the directional derivatives. Since, as seen from (6), the directional derivatives $df(\bar{x},\xi)$ and $dg(\bar{x},\xi)$, respectively, approximate the values of the functions $f(x)$ and $g(x)$ near $\bar{x}$, we may state that the Lagrangian reducibility is determined by the local structures of the functions at $\bar{x}$. It will turn out in the following sections that not only the first order derivatives but also their higher order derivatives need to be considered to investigate the local structures in question.

IV. A Necessary and Sufficient Condition Ensuring the Lagrangian Reduction

1. In the present and the following sections, we shall seek conditions ensuring constrained maximum problems to be reducible. In this section, we shall deal with conditions involving the objective function $f(x)$ and the constraint functions $g(x)$.

It is natural to expect that the reducibility depends upon the

properties of both f(x) and g(x), since the Lagrangian form (21) depends on both. That this is in fact so may be seen from example (24) and the following examples: let m=n=1 and

(29) $f(x) = -x^2$,

$g(x) = -x^2$; then

$L(x,y) = -(1+y)x^2$;

also, let m=n=1, and

(30) $f(x) = x$,

$g(x) = -|x|$; then

$$L(x,y) = x - y|x| = \begin{cases} (1-y)x, & \text{if } x \geq 0, \\ (1+y)x, & \text{if } x \leq 0. \end{cases}$$

All the functions appearing in (29) and (30) are concave. It is easily seen that $\bar{x} = 0$ is an optimum and $(\bar{x},\bar{y}) = (0,1)$ is a saddle-point for each of the two problems; i.e., they are both Lagrange-reducible. Examples (24) and (29) show that in general the Lagrange reducibility depends on f(x), and examples (24) and (30) show that it depends on g(x).

2. The following LEMMA will be used later for proving theorem 2, but the LEMMA itself gives a sufficient condition for f(x) and g(x) to be Lagrange-reducible at $\bar{x}$:

LEMMA 6: If $dg_E(\bar{x},\xi) \geq 0$ with $\xi \neq 0$ implies $df(\bar{x},\xi) < 0$, then there exists $y_E \geq 0$ for which (27) holds.

Proof: In the first place, define

(31) $$A = \left\{ \begin{pmatrix} a_0 \\ a \end{pmatrix} \;\middle|\; \begin{pmatrix} a_0 \\ a \end{pmatrix} \leqq \begin{pmatrix} df(\bar{x};\xi) \\ dg_E(\bar{x};\xi) \end{pmatrix}, \text{ for some } \xi \right\}.$$

It is immediately seen from (7) and (8) that A is a convex cone.

Assume that $dg_E(\xi) \geqq 0$ with $\xi \neq 0$ implies $df(\xi) < 0$. Let c_0 be a positive number. Then by the definition of A, we have $(c_0,0) \notin A$. We shall show that there exists a neighborhood, say S, of the point $(c_0,0)$ such that $S \cap A = \emptyset$. Suppose not. Then there exists a sequence, say $\{(c_0^q,c^q)\}$, converging to $(c^0,0)$ as $q \to \infty$ such that $(c_0^q,c^q) \in A$ for all q. By the definition of A, there exists a sequence, say $\{\eta^q\}$ such that

$$\begin{pmatrix} c_0^q \\ c^q \end{pmatrix} \leqq \begin{pmatrix} df(\eta^q) \\ dg_E(\eta^q) \end{pmatrix}, \text{ for all q.}$$

Since $c_0 > 0$, we may assume that the sequence is chosen in such a way that $c_0^q > 0$. Then, we have $df(\eta^q) > 0$ so that by (8), $\eta^q \neq 0$.

Let $$\lambda^q = |\eta^q| > 0, \text{ and}$$
$$\xi^q = \eta^q/\lambda^q.$$

Then,

(32) $$|\xi^q| = 1, \text{ and}$$

(33) $$\begin{pmatrix} c_0^q \\ c^q \end{pmatrix} \leqq \lambda^q \begin{pmatrix} df(\xi^q) \\ dg_E(\xi^q) \end{pmatrix}, \text{ for all q.}$$

From (32) we may assume that the sequence $\{\xi^q\}$ is chosen so that $\xi^q \to \xi^0 \neq 0$, say. Suppose for the moment that $\{\lambda^q\}$ is bounded.

Then we may further assume that the sequences $\{\xi^q\}$ and $\{\lambda^q\}$ are chosen so that $\lambda^q \to \lambda^\circ \geqq 0$, say. Then, passing to the limit of (33), we obtain

$$\begin{pmatrix} c_0 \\ 0 \end{pmatrix} \leqq \lambda^\circ \begin{pmatrix} df(\xi^\circ) \\ dg_E(\xi^\circ) \end{pmatrix}. \tag{34}$$

Since $c_0 > 0$, it follows that $\lambda^\circ > 0$ and $df(\xi^\circ) > 0$. But this contradicts to the assumption of the lemma, since (34) implies $dg_E(\xi^\circ) \geqq 0$. Hence $\{\lambda^q\}$ is not bounded; i.e., $\lambda^q \to +\infty$ as $q \to +\infty$. Since we have from (33)

$$\begin{pmatrix} c_0^q/\lambda^q \\ c^q/\lambda^q \end{pmatrix} \leqq \begin{pmatrix} df(\xi^q) \\ dg_E(\xi^q) \end{pmatrix}, \text{ for all } q,$$

it follows that $df(\xi^\circ) \geqq 0$ and $dg_E(\xi^\circ) \geqq 0$. But this also contradicts to the assumption of the lemma, since $\xi^\circ \neq 0$. Therefore, we may assert that there exists a neighborhood S of the point $(c_0,0)$, such that $S \cap A = \emptyset$.

Let us now choose a point $(c_0,c) \in S$ such that $c_0 > 0$ and $c < 0$. Clearly $(c_0,c) \notin A$. Since A is a cone, one has

$$\lambda(c_0,c) \in A, \quad \text{for any } \lambda > 0. \tag{35}$$

Next we define

$$B = \left\{ \begin{pmatrix} b_0 \\ b \end{pmatrix} \middle| \begin{pmatrix} b_0 \\ b \end{pmatrix} > \lambda \begin{pmatrix} c_0 \\ c \end{pmatrix}, \text{ for some } \lambda > 0 \right\}.$$

Clearly, B is a convex set. Further, it can be shown that $A \cap B = \emptyset$. For, if not, there would exist a point $(b_0,b) \in A$ such that

$(b_0,b) > \lambda(c_0,c)$ for some $\lambda > 0$. But then (31) implies $\lambda(c_0,c)\varepsilon A$, which is a contradiction to (35).

The sets A and B are convex and possess no point in common. Therefore by a separation theorem of convex sets, there exists a nonvanishing vector, say $(v_0,v) \neq 0$, such that

(36) $$v_0a_0 + va \leqq v_0b_0 + vb, \quad \text{for all } (a_0,a)\varepsilon A \text{ and } (b_0,b)\varepsilon B.$$

Since $0\varepsilon A$, (36) implies that

(37) $$v_0b_0 + vb \geqq 0, \quad \text{for all } (b_0,b)\varepsilon B.$$

Next we observe that the definition of B implies that if $(b_0,b) > 0$, then $(b_0,b)\varepsilon B$. Hence we must have $(v_0,v) \geqq 0$, but since $(v_0,v) \neq 0$, it follows that $(v_0,v) \geq 0$.[12/] This implies that if $v = 0$, then $v_0 > 0$. On the other hand, if $v \geq 0$, then we have $vc < 0$, and hence

$$v_0 \geqq -vc/c_0 > 0,$$

the first inequality above being obtained from (37) and the fact that (c_0,c) is in the closure of B. Summing up, we obtain $v_0 > 0$ and $v \geqq 0$.

Now define $y_E = v/v_0 \geqq 0$. Since 0 is in the closure of B, (36) in addition to (30) yields

(27) $$df(\bar{x};\xi) + y_E dg_E(\bar{x};\xi) \leqq 0, \quad \text{for all } \xi. \qquad \text{(QED)}$$

3. Next let us proceed to state a condition, which we shall prove is necessary and sufficient for the Lagrangian reduction to be valid. The condition will be stated below recursively with some auxiliary definitions.

Let k be a positive integer (k will be specified later by (45)).

For each i ($1 \leqq i \leqq k$), define recursively (38) through (45):

$$(38) \qquad X^i = \begin{cases} R^n, & \text{if } i = 1, \\ (\wedge^{i-1})^{\perp}, & \text{if } 1 < i \leqq k, \end{cases}$$

where R^n is the n-space, $(\wedge^{i-1})^{\perp} = \{\xi \in X^{i-1} \mid \xi\eta = 0 \text{ for all } \eta \in \wedge^{i-1}\}$ is the orthogonal complement of $\wedge^{i-1}$ with respect to X^{i-1}, and $\wedge^{i-1}$ will be defined later by (44). Let

$$(39) \qquad L^i = \{\xi \in X^i \mid d^i g_{E^i}(\bar{x}, \bar{\xi}^{(i-1)}; \xi) \geqq 0\} \text{ and}$$

$$(40) \qquad N^i = \{\xi \in L^i \mid d\ f(\bar{x}, \bar{\xi}^{(i-1)}; \xi) \geqq 0\}, \qquad (i=1,\ldots,k),$$

where

$$(41) \qquad E^{i+1} = \begin{cases} E, & \text{if } i = 0, \\ \{j \in E^i \mid d^i g_j(\bar{x}, \bar{\xi}^{(i-1)}; \xi) = 0 \text{ for all } \xi \in N^i\}, & \\ & \text{if } 1 \leqq i < k; \end{cases}$$

$$(42) \qquad F^{i+1} = \begin{cases} F, & \text{if } i = 0, \\ E^i - E^{i+1}, & \text{if } 1 \leqq i < k; \end{cases}$$

$$(43) \qquad \bar{\xi}^{(i)} = (\bar{\xi}^1, \ldots, \bar{\xi}^i), \text{ and}$$

ξ_i is a relative interior point of N^i, which is a convex cone as shown below (i=1,...k-1).

Further, we define

$$(44) \qquad \wedge^i = \text{the subspace of } R^n \text{ spanned by } N^i (i=1,\ldots,k-1);$$

$$(45) \qquad k = \min \{i > 0 \mid N^i = \{0\}\}.$$

We now state:

REGULARITY CONDITION:

(46) $d^i g_{Ei}(\bar{x},\bar{\xi}^{(i-1)};\xi) \geqq 0$ with $\xi \varepsilon X^i$

implies

$$d^i f(\bar{x},\bar{\xi}^{(i-1)};\xi) \leqq 0, \quad \text{for } i=1,\ldots,k,$$

where $\bar{\xi}^{(i)}$ $(i=1,\ldots k-1)$ satisfy (43).

In order to verify that (38) through (45) are well defined so that (46) is a meaningful condition, let us first observe from (38) that

(47) $X^i = \wedge^i \oplus X^{i+1}$ $(i=1,\ldots,k-1)$,

and hence that

(48) $R^n = X^1 = \wedge^1 \oplus \ldots \oplus \wedge^{h-1} \oplus X^h$ $(h=1,\ldots,k)$,

where the symbol $\oplus$ indicates the direct-sum of orthogonal subspaces. In view of (47) one has $\dim X^{i+1} = \dim X^i - \dim \wedge^i$, and hence from (44)

(49) $\dim X^{i+1} < \dim X^i$, unless $N^i = \{0\}$.

Therefore, $N^i = \{0\}$ for some (finite) i so that k satisfying (45) is determined. It is also seen from (40) and (49) that $k \leqq n+1$.

Next, (25), (41), and (42) imply that

(50) $E^0 \supset E^1 \supset E^2 \supset \ldots \supset E^k$,

(51) $E^{i-1} = E^i \cup F^i$ $(i=1,\ldots,k)$,

where $E^0 = \{1,\ldots,m\}$ and the symbol $\cup$ indicates the union of disjoint sets. Furthermore,

(52) $E^h = F^{h+1} \cup \ldots \cup F^{h'} \cup E^{h'}$

$(h < h';\ h = 0,1,\ldots,k-1;\ h'=1,\ldots,k)$.

In particular (put h = 0),

(53) $$E^0 = F^1 \textcircled{\cup} \ldots . \textcircled{\cup} F^{h'} \textcircled{\cup} E^{h'} \qquad (h'=1,\ldots,k),$$

It is easily seen from (7) and (8) applied to the functions $d^i g_{Ei}(\xi)$ and $d^i f(\xi)$ that the sets N^i and L^i are closed convex cones (i=1,...k). Therefore, one can always choose a vector $\bar{\xi}^i$ which is a relative interior point of the convex cone N^i (i=1,...,k-1), since the relative interior of a convex cone is never empty.[13/]

This completes showing that (38) through (45) is a meaningful construction.

4. The following lemma can be proved immediately in view of (40):

LEMMA 7: The regularity condition (46) implies that

$$d^i f(\bar{x},\bar{\xi}^{(i-1)};\xi) = 0, \quad \text{for all } \xi \epsilon N^i \qquad (i=1,\ldots,k).$$

LEMMA 8: If (46) holds, then

(54) $$d^{i+1} f(\bar{x},\bar{\xi}^{(i)};\xi) = 0, \quad \text{and}$$

$$d^{i+1} g_{Ei+1}(\bar{x},\bar{\xi}^{(i)};\xi) = 0, \quad \text{for all } \xi \epsilon \wedge^i,$$

where $\bar{\xi}^{(i)}$ satisfies (43) (i=1,...,k-1).

Proof: Since $\xi \epsilon \wedge^i$ and $\bar{\xi}^i$ is a relative interior point of N^i, one may find in view of (44) a positive number, say μ', such that $\bar{\xi}^i + \mu\xi \epsilon N^i$ for all $0 < \mu < \mu'$. Then (6) yields

$$d^i f(\bar{\xi}^i + \mu\xi) = d^i f(\bar{\xi}^i) + d^{i+1} f(\bar{\xi}^i;\xi)\mu + o(\mu).$$

But $d^i f(\bar{\xi}^i) = 0$ and $d^i f(\bar{\xi}^i + \mu\xi) = 0$ if $0 < \mu < \mu'$ by LEMMA 7. Hence,

$$d^{i+1} f(\bar{\xi}^i;\xi) = -o(\mu)/\mu, \quad \text{for all } 0 < \mu < \mu'.$$

Passing to the limit, one obtains the first equation of (54).

The second equation of (54) may be proved similarly; we have only to observe from (41) and (51) that if $\xi \varepsilon \wedge^i$, then

$$d^i g_{Ei+1}(\bar{\xi}^i) = 0, \text{ and}$$

$$d^i g_{Ei+1}(\bar{\xi}^i + \mu\xi) = 0, \text{ for all } \mu \text{ sufficiently small. (QED)}$$

LEMMA 9: Let ξ be an arbitrary vector in R^n, and let

$$\xi = \sum_{i=1}^{h-1} \xi^i + \hat{\xi}^h, \qquad (h=1,\ldots,k), \tag{55}$$

where

$$\begin{aligned} &\xi^i \varepsilon \wedge^i \qquad (i=1,\ldots k-1), \\ &\hat{\xi}^h \varepsilon X^h \qquad (h=1,\ldots,k). \end{aligned} \tag{56}$$

(Such decomposition of ξ is always possible and unique by virtue of (48).)

If (46) is satisfied, then

$$d^i f(\bar{\xi}^{(i-1)}; \hat{\xi}^i) \leqq d^{i+1} f(\bar{\xi}^{(i)}; \hat{\xi}^{i+1}), \text{ and} \tag{57}$$

$$d^i g_{Ei+1}(\bar{\xi}^{(i-1)}; \hat{\xi}^i) \leqq d^{i+1} g_{Ei+1}(\bar{\xi}^i; \hat{\xi}^{i+1}), \text{ for } i=1,\ldots,k-1,$$

where $\bar{\xi}^{(i-1)}$ and $\bar{\xi}^{(i)}$ are chosen so as to satisfy (43). In particular (since $\hat{\xi}^1 = \xi$),

$$df(\xi) \leqq d^k f(\bar{\xi}^{(k-1)}; \hat{\xi}^k), \text{ and} \tag{58}$$

$$dg_{Ek}(\xi) \leqq d^k g_{Ek}(\bar{\xi}^{(k-1)}; \hat{\xi}^k).$$

Proof: From (55), we have

$$\hat{\xi}^i = \xi^i + \hat{\xi}^{i+1}. \text{ Then,} \tag{59}$$

$$d^{i+1}f(\bar{\xi}^i;\hat{\xi}^{i+1}) = d^{i+1}f(\bar{\xi}^i;\hat{\xi}^i - \xi^i) \qquad \text{(by (59))}$$

$$\geq d^{i+1}f(\bar{\xi}^i;\hat{\xi}^i) + d^{i+1}f(\bar{\xi}^i;-\xi^i) \qquad \text{(by (8))}$$

$$= d^{i+1}f(\bar{\xi}^i;\hat{\xi}^i) \qquad \text{(by (56) and lemma 8)}$$

$$\geq d^i f(\hat{\xi}^i) \qquad \text{(by lemma 1 and (16)).}$$

The second inequality of (57) may be proved in a similar way. (QED)

THEOREM 2: (a) If the functions $f(x)$ and $g(x)$ are Lagrange reducible at $\bar{x}$, then the regularity condition (46) holds at $\bar{x}$ for <u>any</u> $\bar{\xi}^i (i=1,\ldots,k-1)$ satisfying (43). Conversely, (b) if the regularity condition (46) holds at $\bar{x}$ for <u>some</u> $\bar{\xi}^i (i=1,\ldots,k-1)$ satisfying (43), then $\bar{x}$ is an optimum and the functions $f(x)$ and $g(x)$ are Lagrange reducible at $\bar{x}$.

Proof: (a) Assume that $f(x)$ and $g(x)$ are Lagrange reducible at $\bar{x}$. Then, by LEMMA 5, there exists $y_E \geq 0$ such that

(27) $\quad df(\xi) + y_E dg_E(\xi) \leq 0, \quad$ for all $\xi \in R^n = X^1$.

Let us agree that the components of y_E and $dg_E(\xi)$ are arranged in the following way (see (52) and put $h'=k$):

(60) $\quad dg_{Eh} = (dg_{Fh+1},\ldots,dg_{Fk}, dg_{Ek})$, and

$$y_{Eh} = (y_{Fh+1}\cdots,y_{Fk}, y_{Ek}), \quad (h=1,\ldots,k).$$

We shall show that (27) implies

(61) $\quad d\, f(\bar{\xi}^{(i-1)};\xi) + y_{Ei}\cdot d^i g_{Ei}(\bar{\xi}^{(i-1)};\xi) \leq 0$, for all $\xi \in X^i$, $(i=1,\ldots k,)$,

where $\bar{\xi}^{(i-1)}$ satisfies (43). Then, since $y_{Ei} \geq 0$ for all i, the

regularity condition (46) follows readily from (61).

We prove (61) from (27) by induction. In the first place, when i=1, then (61) coincides with (27). Next, assume that (61) holds for i. Then, by (51),

$$d^i f(\xi) + y_{Ei+1} \cdot d^i g_{Ei+1}(\xi) + y_{Fi+1} \cdot d^i g_{Fi+1}(\xi) \leqq 0, \tag{62}$$

for all $\xi \in X^i$.

By (41) and LEMMA 7 with the induction hypothesis,

$$d^i f(\xi) = 0, \text{ and} \tag{63}$$

$$d^i g_{Ei+1}(\xi) = 0, \quad \text{for all } \xi \in N^i.$$

Hence $y_{Fi+1} \cdot dg_{Fi+1}(\xi) \leqq 0$, for all $\xi \in N^i$. On the other hand, $y_{Fi+1} \geqq 0$ and $d^i g_{Fi+1}(\xi) \geqq 0$ for all $\xi \in N^i$ by (40). Hence $y_j dg_j(\xi) = 0$ for all $j \in F^{i+1}$ and $\xi \in N^i$. But by the definition of F^{i+1}, for each $j \in F^{i+1}$ there exists $\xi \in N^i$ such that $dg_j(\xi) > 0$. Hence $y_j = 0$, for each $j \in F^{i+1}$. I.e., $y_{Fi+1} = 0$, so that from (62)

$$d^i f(\xi) + y_{Ei+1} \cdot d^i g_{Ei+1}(\xi) \leqq 0, \text{ for all } \xi \in X^i. \tag{64}$$

Let $\bar{\xi}^i$ be an arbitrary relative interior point of N^i. Then, for all $\mu > 0$ and $\xi \in X^{i+1}$, we have $(\bar{\xi}^i + \mu\xi) \in X^i$ (see (44) and (47)), and hence by (64)

$$d^i f(\bar{\xi}^i + \mu\xi) + y_{Ei+1} \cdot d^i g_{Ei+1}(\bar{\xi}^i + \mu\xi) \leqq 0.$$

In view of (6) and (63), we get

$$d^{i+1} f(\xi) + y_{Ei+1} \cdot d^{i+1} g_{Ei+1}(\xi) \leqq -o(\mu)/\mu, \text{ for all } \mu > 0$$

and $\xi \in X^{i+1}$,

from which the relation (61) with (i) replaced by (i+1) is derived. This completes the induction, proving (61) for all i.

(b) Assume that the regularity condition (46) holds for some $\bar{\xi}^i$ (i=1,...,k-1) satisfying (43). By (45), $N^k = \{0\}$. Therefore, $d^k g_{Ek}(\bar{x}, \xi^{(k-1)}; \xi) \geq 0$ with $\xi \in X^k$ and $\xi \neq 0$ implies $d^k f(\bar{x}, \xi^{(k-1)}; \xi) < 0$. Hence, by applying LEMMA 6 to the functions $d^k f(\xi)$ and $d^k g_{Ek}(\xi)$, we have

(65) there exists $y_{Ek} \geq 0$ such that

$$d^k f(\bar{x}, \bar{\xi}^{(k-1)}; \xi) + y_{Ek} . d^k g_{Ek}(\bar{x}, \bar{\xi}^{(k-1)}; \xi) \leq 0, \text{ for all } \xi \in X^k.$$

But (65) with (58) of lemma 9 implies that

(66) $$df(\bar{x}; \xi) + y_{Ek} . dg_{Ek}(\bar{x}; \xi) \leq 0, \text{ for all } \xi \in R^n.$$

Define

(67) $$y_E = (0, \ldots, 0, y_{Ek}).$$

From (66) and (67) we obtain (27). Hence, $(\bar{x}, \bar{y}) = (\bar{x}, (y_E, 0))$ is a saddle-point of the Lagrangian form (21). By Uzawa [6, Theorem 1], $\bar{x}$ is an optimum. Therefore, f(x) and g(x) are Lagrange reducible at $\bar{x}$. (QED)

For example (24), we compute $df(\bar{x}, \xi) = \xi$, $dg(\bar{x}, \xi) = 0$, where $\bar{x} = 0$. Hence, $L^1 = R^1$, $N^1 = [0, \infty)$, $L^2 = \{0\}$, $N^2 = \{0\}$, and k = 2. The condition (46) is not satisfied for i = 1 so that the reduction impossible.

For example (29), we have $df(\bar{x},\xi) = dg(\bar{x},\xi) = 0$ and $\bar{x} = 0$; hence $L^1 = N^1 = R^1$ and $L^2 = N^2 = \{0\}$, and $k = 2$. The condition (46) is satisfied for $i = 1, 2$.

For example (30), we have $df(\bar{x},\xi) = \xi$, $dg(\bar{x},\xi) = -\xi$, $x = 0$, $L^1 = \{0\}$, $N^1 = \{0\}$, and $k = 1$. This satisfies (46).

V. Constraint Qualifications

1. In the present section, we shall be concerned with regularity conditions stated in terms of the constraint functions $g(x)$ only.

LAGRANGIAN REGULARITY: The constraint functions $g(x)$ are <u>Lagrange regular</u> at $\bar{x}$, if the Lagrangian reduction is possible for any concave function $f(x)$ at $\bar{x}$.

Conditions on $g(x)$ ensuring the Lagrangian regularity are called <u>constraint qualifications.</u> Since Kuhn and Tucker [3], a number of constraint qualifications has been proposed. When the constraint functions $g(x)$ are assumed to be concave but not necessarily differentiable, however, Slater-Uzawa's conditions, which will be stated later (see Corollaries 2 and 3 of Theorem 3), seem to be the only known constraint qualifications. In what follows, we shall present a fairly weak constraint qualification, from which Slater-Uzawa's conditions can be derived. In addition, another condition sufficient for the Lagrangian regularity will be derived from it (see Corollary 1 of Theorem 3).

2. In the first place, we need to state some definitions. For the

given constraint functions $g(x)$ and the given vector $\bar{x}$, we apply the recursive definitions (38) through (45) with the following modification: retain (38) through (45) but delete (40) defining closed convex cones $N^i (i=1,\ldots,k)$ and replace it by: (40a) N^i is a closed convex cone contained in $L^i (i=1,\ldots,k)$.

Since all the definitions (38) through (45) except (40) are stated in the form independent of the function $f(x)$, the above construction is meaningful at least formally. Further, it can easily be verified that the statements (47) through the one preceding Lemma 7 still remain true after (40) is replaced by (40a). Therefore, we may assert that the above construction is meaningful.

Let us further define:

(68) $$K^i = \text{the closure of } \left\{\xi \epsilon X^i \,\middle|\, g_{E^i}(x + \sum_{r=1}^{i-1} (\mu^1 \ldots \mu^r)\bar{\xi}^r + (\mu^1 \ldots \mu^i)\xi \geq 0, \text{ for some } \mu^r > 0 \ (r=1,\ldots,i)\right\}$$

where $\bar{\xi}^r (r=1,\ldots,i-1)$ satisfy (43). It is seen that $K^1 = K$, where K is defined previously by (20).

LEMMA 10: For any N^i satisfying (40a) and for any $\bar{\xi}^i$ satisfying (43), $K^i \subset L^i (i=1,\ldots,k)$.

Proof: Assume $\xi \epsilon K^i$. By (68) and Lemma 2(a), there exists a sequence $\{\xi^q\}$ converging to ξ as $q \to \infty$ such that for each q

$$g_{Ei}(\bar{x}) + \sum_{r=1}^{i-1} d^r g_{Ei}(\bar{x},\bar{\xi}^{(r-1)};\bar{\xi}^r)(\mu^1\ldots\mu^r) + d^i g_{Ei}(\bar{x},\bar{\xi}^{(i-1)};\xi^q)(\mu^1\ldots\mu^i)$$

$$+ \sum_{r=1}^{i} o^r(\mu^r)(\mu^1\ldots\mu^{r-1}) \geq 0, \text{ for some } \mu^r > 0 \ (r=1,\ldots,i).$$

But by the definition of E^i, the first two terms of the above expression vanish, so that for each q

$$d^i g_{Ei}(\bar{x},\bar{\xi}^{(i-1)};\xi^q)(\mu^1\ldots\mu^i) \geq - \sum_{r=1}^{i} o^r(\mu^r)(\mu^1\ldots\mu^{r-1}) \geq 0, \text{ for}$$

$$\text{some } \mu^r > 0 \ (r=1,\ldots i).$$

Hence, by Lemma 2(b),

$$d^i g_{Ei}(\bar{x},\bar{\xi}^{(i-1)};\xi^q) \geq 0, \text{ for each q,}$$

which implies that

$$d^i g_{Ei}(\bar{x},\bar{\xi}^{(i-1)};\xi) \geq 0;$$

i.e., $\xi \in L^i$. (QED)

CONSTRAINT QUALIFICATION:

(69) For <u>any</u> N^i satisfying (40a) and for <u>some</u> $\bar{\xi}^i$ satisfying (43), $L^i \subset K^i$ (i=1,...,k).

In view of Lemma 10, one knows that (69) is equivalent to $L^i = K^i$ (i=1,...,k). Further, for a special case where the functions g(x) are differentiable, (69) reduces to Constraint Qualfication W in [1]. We shall prove later that (69) is sufficient for g(x) to be Lagrange regular at $\bar{x}$.

LEMMA 11: If $F^{i+1} \neq \emptyset$, then

$$(70) \qquad d^i g_{F^{i+1}}(\bar{x}, \bar{\xi}^{(i-1)}; \bar{\xi}^i) > 0, \text{ for } i=1,\ldots,k,$$

for <u>any</u> N^i satisfying (40a) and for <u>any</u> $\bar{\xi}^i$ satisfying (43).

Proof: Suppose the lemma does not hold; i.e., for some $j \epsilon F^{i+1}$ there exists a $\bar{\xi}^i$ in the relative interior of N^i for which $d^i g_j(\bar{\xi}^i) = 0$. Let ξ be an arbitrary point of N^i. Then, since $\bar{\xi}^i$ is in the relative interior of N^i, there exists $t(0 < t < 1)$ such that $(\bar{\xi}^i - t\xi) \epsilon N^i$. Since $\bar{\xi}^i = t\xi + (\bar{\xi}^i - t\xi)$, we get in view of (8)

$$(71) \qquad 0 = d^i g_j(\bar{\xi}^i) \geq t d^i g_j(\xi) + (1-t) d^i g_j(\bar{\xi}^i - t\xi).$$

Since $d^i g_j(\xi) \geq 0$ and $d^i g_j(\bar{\xi}^i - t\xi) \geq 0$ by (39) and (40a), we obtain

$$(72) \qquad d^i g_j(\xi) = 0, \text{ for all } \xi \epsilon N^i.$$

But this contradicts to the definition of F^{i+1}. (QED)

LEMMA 12: For <u>any</u> N^i satisfying (40a) and for <u>any</u> $\bar{\xi}^i$ satisfying (43),

$$(73) \qquad K^i = \text{the closure of}$$

$$\left\{ \xi \epsilon X^i \,\middle|\, g(\bar{x} + \sum_{r=1}^{i-1} (\mu^1 \ldots \mu^r)\bar{\xi}^r + (\mu^1 \ldots \mu^i)\xi) \geq 0, \text{ for some } \mu^r > 0 \ (r=1,\ldots,i) \right\}, \text{ for } i=1,\ldots,k.$$

(For a proof, see the appendix.)

THEOREM 3: If the constraint functions $g(x)$ satisfy Constraint Qualification (69) at $\bar{x}$, then they are Lagrange regular at $\bar{x}$.

Proof: Let $f(x)$ be an arbitrary concave objective function which is maximized at $\bar{x}$ subject to $g(x) \geq 0$. To this $f(x)$ and the given

functions $g(x)$, apply the definitions (38) through (45). The set N^i thus obtained is a closed convex cone contained in $L^i (i=1,\ldots,k)$, and the vector $\bar{\xi}^i$ is in the relative interior of $N^i (i=1,\ldots,k-1)$. Therefore, by the assumption of the theorem, $L^i \subset K^i$ for $i=1,\ldots,k$.

We shall show that the regularity condition (46) is satisfied by $f(x)$ and $g(x)$ at $\bar{x}$. The proof will be carried out inductively.

For $i=1$, assume $dg_{E1}(\bar{x},\xi) \geq 0$; i.e., $\xi \epsilon L^1$. Then (69) implies that $\xi \epsilon K^1 = K$. In view of Theorem 1, one obtains $df(\bar{x},\xi) \leq 0$. Thus, (46) is satisfied for $i = 1$.

Next assume that (46) is satisfied for all r less than i. Let $d^i g_{Ei}(\bar{x},\bar{\xi}^{(i-1)};\xi) \geq 0$ and $\xi \epsilon X^i$; i.e., $\xi \epsilon L^i$. By (69) we have $\xi \epsilon K^i$. Lemma 12 then implies that there exists a sequence $\{\xi^q\}$ converging to ξ as $q \to \infty$ such that for each q

$$g(\bar{x} + \sum_{r=1}^{i-1} (\mu^1 \ldots \mu^r)\bar{\xi}^r + (\mu^1 \ldots \mu^i)\xi^q) \geq 0, \text{ for some } \mu^r > 0 \ (r=1,\ldots,i).$$

Since $\bar{\xi}^r \epsilon N^r \subset L^r$ for all r less than i, the induction hypothesis and (69) imply for $r=1,\ldots,i-1$,

$$g(\bar{x} + \sum_{s=1}^{r} (\mu^{1r} \ldots \mu^{sr})\bar{\xi}^s) \geq 0, \text{ for some } \mu^{sr} > 0 \ (s=1,\ldots,r).$$

Then, in view of Lemma 3, we have

$$g(\bar{x} + \sum_{r=1}^{i-1} (\lambda^1 \ldots \lambda^r)\bar{\xi}^r + (\lambda^1 \ldots \lambda^i)\xi^q) \geq 0,$$

for <u>any</u> $\lambda^r > 0$ sufficiently small $(r=1,\ldots,i)$.

Since $\bar{x}$ is an optimum, we have from Lemma 2(a) that, for each q and for each $\lambda^r > 0$ sufficiently small (r=1,...,i),

$$f(\bar{x}) \geq f(\bar{x} + \sum_{r=1}^{i-1} (\lambda^1 \ldots \lambda^r)\bar{\xi}^r + (\lambda^1 \ldots \lambda^i)\xi^q)$$

$$= f(\bar{x}) + \sum_{r=1}^{i-1} d^r f(\bar{x}, \bar{\xi}^{(r-1)}; \bar{\xi}^r)(\lambda^1 \ldots \lambda^r)$$

$$+ d^i f(\bar{x}, \bar{\xi}^{(i-1)}; \xi^q)(\lambda^1 \ldots \lambda^i) + \sum_{r=1}^{i} o^r(\lambda^r)(\lambda^1 \ldots \lambda^{r-1}).$$

By the induction hypothesis and Lemma 7,

$$d^r f(\bar{x}, \bar{\xi}^{(r-1)}; \bar{\xi}^r) = 0, \text{ for } r=1,\ldots,i-1.$$

Hence, for each q,

$$d^i f(\bar{x}, \bar{\xi}^{(i-1)}; \xi^q)(\lambda^1 \ldots \lambda^i) \leq - \sum_{r=1}^{i} o^r(\lambda^r)(\lambda^1 \ldots \lambda^{r-1}),$$

for any $\lambda^r > 0$ sufficiently small (r=1,...,r).

Lemma 2(b) then implies that for each q

$$d^i f(\bar{x}, \bar{\xi}^{(i-1)}; \xi^q) \leq 0, \text{ and hence}$$

$$d^i f(\bar{x}, \bar{\xi}^{(i-1)}; \xi) \leq 0.$$

I.e., (46) is satisfied for i. This completes the induction, proving the theorem. (QED)

Corollary 1: If, for any $\xi \neq 0$, there exists $j \in E^1$ such that $dg_j(\bar{x}, \xi) < 0$, then g(x) is Lagrange regular at $\bar{x}$.

Proof: Under the given condition, we have

$$K^1 = L^1 = N^1 = \{0\},$$

with k = 1. Hence, (69) is satisfied at $\bar{x}$. (QED)

Corollary 2 (Uzawa): Let G^1 and H^1 be, respectively, the set of linear constraints and the set of other constraints in E^1. If there exists $\xi \in R^n$ such that

$$(74) \qquad dg_{G1}(\bar{x};\xi) \geqq 0 \text{ and}$$
$$dg_{H1}(\bar{x};\xi) > 0,$$

then $g(x)$ is Lagrange regular at $\bar{x}$.

Proof: In the first place, we apply the definitions (38) through (45) to the given functions $g(x)$ and the given vector $\bar{x}$ where (40) is replaced by (40a).

Define

$$(75) \qquad G^i = \left\{ j \in E^i \,\middle|\, g_j(x) \text{ is linear} \right\}, \text{ and}$$
$$H^i = E^i - G^i, \quad \text{for } i=1,\ldots,k.$$

Then, obviously (see (50)),

$$(76) \qquad G^1 \supset G^2 \supset \ldots \supset G^k, \text{ and}$$
$$E^i \supset G^i, \; E^i \supset H^i \; (i=1,\ldots,k).$$

For any N^i satisfying (40a) and for any $\bar{\xi}^i$ satisfying (43), the second half of (57) in Lemma 9 still holds. Therefore, in view of (74), (57), and (75), we may assert that there exist $\xi^i \in A^i$ and $\hat{\xi}^i \in X^i$ such that

$$(77) \qquad d^i g_{Hi}(\hat{\xi}^i) > 0, \text{ for } i=1,\ldots,k,$$

where ξ^i and $\hat{\xi}^i$ satisfy (55) and (56). (Note that $\hat{\xi}^1 = \xi$.)

Next, let us write the linear constraints as

(78) $g_{Gi}(x) = A_{Gi}(x-\bar{x})$, for i=1,...,k,

where A_{Gi}(i=1,...,k) are appropriate matrices. By (13),

(79) $d^i g_{Gh}(\bar{x},\bar{\xi}^{(i-1)};\xi) = A_{gh}\xi$, for i,h=1,...,k.

Now from (74), (55), and (79),

(80) $$0 \leqq dg_{Gi}(\xi) = A_{Gi}\xi$$
$$= \sum_{r=1}^{i-1} A_{Gi}\xi^r + A_{Gi}\hat{\xi}^i,$$
$$= \sum_{r=1}^{i-1} d^{r+1} g_{Gi}(\xi^r) + A_{Gi}\hat{\xi}^i, \text{ for } i=1,...,k.$$

Then Lemma 8 and (56) yield

(81) $A_{Gi}\hat{\xi}^i \geqq 0$, for i=1,...,k.

Let us now choose an arbitrary $\tilde{\xi}^i \in L^i$ for each i; i.e.,

(82) $d^i g_{Ei}(\tilde{\xi}^i) \geqq 0$, or

$$d^i g_{Gi}(\tilde{\xi}^i) = A_{Gi}\tilde{\xi}^i \geqq 0 \quad \text{and} \quad d^i g_{Hi}(\tilde{\xi}^i) \geqq 0.$$

Define

(83) $\xi^i(t) = t\hat{\xi}^i + (1-t)\tilde{\xi}^i$,

where $0 < t < 1$. Then, by Lemma 2(a), (76), (41), (43), and (8),

(84) $$g_{Hi}(\bar{x} + \sum_{r=1}^{i-1} (\mu^1..\mu^r)\bar{\xi}^r + (\mu^1..\mu^r)\xi(t)) = d^i g_{Hi}(\xi(t))(\mu^1..\mu^r) + \sum_{r=1}^{i} o^r(\mu^r)(\mu^1..\mu^{r-1})$$
$$\geqq [td^i g_{Hi}(\hat{\xi}^i) + (1-t)d^i g_{Hi}(\tilde{\xi}^i)](\mu^1..\mu^r) + \sum_{r=1}^{i} o^r(\mu^r)(\mu^1..\mu^{r-1}) > 0,$$

for some $\mu^r > 0$ (r=1,...,i),

the last inequality being obtained from (77), (82), and Lemma 2(b).

On the other hand, we get from (78), (76), (41), (43), and (83)

$$(85)\quad g_{Gi}(\bar{x} + \sum_{r=1}^{i-1} (\mu^1 . . \mu^r)\bar{\xi}^r + (\mu^1 . . \mu^i)\xi^i(t)) = A_{Gi}\Big\{ \sum_{r=1}^{i-1} (\mu^1 . . \mu^r)\bar{\xi}^r + (\mu^1 . . \mu^i)\xi^i(t)\Big\}$$

$$= (\mu^1 . . \mu^i)\Big\{tA_{Gi}\hat{\xi}^i + (1-t)A_{Gi}\tilde{\xi}^i\Big\} \geq 0, \text{ for some } \mu^r > 0 \ (r=1,\ldots,i),$$

where the last inequality is obtained from (81) and (82).

Finally, from (84) and (85), we may conclude that $\xi^i(t) \in K^i$ if $0 < t < 1$. Since K^i is closed, $\tilde{\xi}^i \in K^i$. Therefore we have

$$(86)\qquad L^i \subset K^i, \text{ for } i=1,\ldots,k.$$

That is, (69) is satisfied. (QED)

Corollary 3 (Slater-Uzawa): If for any $j \in E^1$ there exists $\xi \in R^n$ such that $dg_j(\bar{x}, \xi) > 0$, then $g(x)$ is Lagrange regular at $\bar{x}$.

Proof: This is a special case of Corollary 2.

(Remark: If the condition in Corollary 1 is satisfied at $\bar{x}$, then for any $x \neq \bar{x}$ there exists $j \in E^1$ such that $g_j(x) < 0$. But the converse is not true. On the other hand, the condition in Corollary 3 holds if and only if there exists x such that, for any j, $g_j(x) > 0$.)

Appendix

[A] Proof of Lemma 2(a):

To begin with, let us spell out the full arguments of the residual functions $o^i(\mu^i)$ appearing in (17):

$$(a1) \qquad o^i(\mu^i) = o^i(\mu^i; x, \xi^{(i-1)}, \sum_{r=i}^{k} (\mu^{i+1}..\mu^r)\xi^r), \text{ for } i=1,\ldots,k.$$

We shall prove Lemma 2(a) by induction on k. In the first place, if k = 1, then it is reduced to (6). Next, assume that (17) with (a1) holds for all k such that $k \leqq s$. By the induction hypothesis where k = 1, we have

$$(a2) \qquad \varphi(x + \sum_{i=1}^{s+1} (\mu^1..\mu^i)\xi^i) = \varphi(x + \mu^1\{\xi^1 + \sum_{i=2}^{s+1} (\mu^2..\mu^i)\xi^i\})$$

$$= \varphi(x) + d\varphi(x; \xi^1 + \sum_{i=2}^{s+1} (\mu^2..\mu^i)\xi^i)\mu^1 + o^1(\mu^1),$$

where

$$(a3) \qquad o^1(\mu^1) = o^1(\mu^1; x, \sum_{i=1}^{s+1} (\mu^2..\mu^i)\xi^i) \leqq 0 \text{ and } o^1(\mu^1)/\mu^1 \uparrow 0 \text{ as } \mu^1 \downarrow 0.$$

Define

$$(a4) \qquad \Psi(\eta) \equiv d\varphi(x; \eta).$$

Since Ψ is a concave function, we may apply to Ψ the induction hypothesis where k = s, obtaining

$$(a5) \quad d\varphi(x, \xi^1 + \sum_{i=2}^{s+1} (\mu^2..\mu^i)\xi^i) = \Psi(\xi^1 + \sum_{i=1}^{s} (\mu^2..\mu^{i+1})\xi^{i+1})$$

$$= \Psi(\xi^1) + \sum_{i=1}^{s} d^i\Psi(\xi^1, \xi^2, \ldots; \xi^{i+1})(\mu^2..\mu^{i+1}) + \sum_{i=1}^{s} p^i(\mu^{i+1})(\mu^2..\mu^i),$$

$$= d\varphi(x, \xi^1) + \sum_{i=2}^{s+1} d^i\varphi(x, \xi^1, \ldots; \xi^i)(\mu^2 \ldots \mu^i)$$

$$+ \sum_{i=2}^{s+1} o^i(\mu^i)(\mu^2 \ldots \mu^{i-1}),$$

where

$$(a6) \quad p^i(\mu^{i+1}) = p^i(\mu^{i+1}; \xi^1, \xi^2, \ldots, \xi^i, \sum_{r=i}^{s} (\mu^{i+2} \ldots \mu^{r+1})\xi^{r+1}) = o^{i+1}(\mu^{i+1})$$

$$= o^{i+1}(\mu^{i+1}; x, \xi^{(i)}, \sum_{r=i+1}^{s+1} (\mu^{i+1} \ldots \mu^r)\xi^r), \text{ and}$$

$$p^i(\mu^{i+1})/\mu^{i+1} = o^{i+1}(\mu^{i+1})/\mu^{i+1} \uparrow 0 \text{ as } \mu^{i+1} \downarrow 0, \quad (i=1,\ldots,s).$$

By combining (a2), (a3), (a5), and (a6), we obtain (17) with (a1) for $k = s+1$. This completes the induction, proving Lemma 2(a) for all k.

[B] Proof of Lemma 2(b):

From (17) and (a1), one may assert:

(b1) For any $\varepsilon > 0$, for each i ($1 \leqq i \leqq k$), and for any $\mu^r > 0$ ($r=i+1,\ldots,k$), one can choose a $\mu^i > 0$ satisfying

$$-\frac{o^i(\mu^i)}{\mu^i} < \frac{\varepsilon}{k} (\mu^{i+1} \ldots \mu^k).$$

Next, choose $\mu^k > 0, \ldots, \mu^1 > 0$ successively in this order so that (b1) is satisfied for $i=k,k-1,\ldots,1$. This is possible since each residual $o^i(\mu^i)$ does not depend on $\mu^1, \ldots, \mu^{i-1}$. For the μ^i's thus obtained, we have from (b1)

$$(b2) \qquad -o^i(\mu^i)(\mu^1 \ldots \mu^{i-1}) < \frac{\varepsilon}{k}(\mu^1 \ldots \mu^k), \text{ for } i=1,\ldots,k.$$

By summing up (b2) over all i, we get (18).

[C] Proof of Lemma 3:

The proof will be conducted by induction on k. Consider first the case where k = 1. From the assumptions $\varphi(x) \geq 0$ and $\varphi(x + \mu^{11}\xi^1) \geq 0$ for some $\mu^{11} > 0$, we get in view of (1)

(c1) $\varphi(x + \lambda^1\xi^1) \geq 0$, for any λ^1 satisfying

$$0 < \lambda^1 < \mu^1 \equiv \min_{\substack{i \\ (r \leq i \leq 1)}} \mu^{ri} (= \mu^{11}) \quad (r=1),$$

which is (19) for k = 1.

Next assume that the lemma holds for each k such that $k \leq s$. Assume further than the condition of the lemma holds for k = s+1; i.e., for i=1,...,s+1,

(c2) $$\varphi(x + \sum_{r=1}^{i} (\mu^{1i}...\mu^{ri})\xi^r) \geq 0, \text{ for some } \mu^{ri} > 0 \ (r=1,...,i).$$

Write (c2) as

(c3) $$\varphi(x + \mu^{1i}\{\sum_{r=1}^{i} (\mu^{2i}...\mu^{ri})\xi^r\}) \geq 0.$$

Then, by applying to (c3) the induction hypothesis for k = 1, we obtain for i=1,...,s+1,

(c4) $$\varphi(x + \lambda^1 \sum_{r=1}^{i} (\mu^{2i}...\mu^{ri})\xi^r) = \varphi((x + \lambda^1\xi^1) + \sum_{r=2}^{i} (\mu^{2i}...\mu^{ri})(\lambda^1\xi^r)) \geq 0,$$

for any λ^1 satisfying $0 < \lambda^1 < \mu^1 \equiv \min_{\substack{i \\ (1 \leq i \leq s+1)}} \mu^{1i}$ and for some $\mu^{ri} > 0$ (r=2,...,i). In particular,

(c5) $\varphi(x + \lambda^1\xi) \geq 0$ for any such λ^1.

By applying again the induction hypothesis for k = s to the inequalities (c4), we obtain, in view of (c5), for i=2,...s+1

$$(c6)\qquad \varphi(x + \lambda^1\xi^1 + \sum_{r=2}^{i} (\lambda^2\ldots\lambda^r)(\lambda^1\xi^r) = \varphi(x + \sum_{r=1}^{i} (\lambda^1\ldots\lambda^r)\xi^r) \geqq 0,$$

for any λ^r satisfying $0 < \lambda^r < \mu^r \equiv \min_{\substack{i \\ (r\leqq i\leqq s+1)}} \mu^{ri}$ $(r=1,\ldots,s+1)$.

By combining (c5) and (c6), we have (19) for k = s+1. This completes the induction, proving the lemma for all k.

[D] Proof of Lemma 12:

In view of (68) and (52) (put h=0 and h'=i), it is sufficient to show that for each i=1,...,k and for each h=1,...,i, $\xi \in K^i$ implies

$$(d1)\qquad g_{Fh}(\bar{x} + \sum_{r=1}^{i-1} (\mu^1\ldots\mu^r)\bar{\xi}^r + (\mu^1\ldots\mu^i)\xi) \geqq 0, \text{ for some } \mu^r > 0 \text{ independent of h } (r=1,\ldots,i).$$

In fact, we shall show that, for any $\xi \in R^n$, (d1) holds strictly.

Let $1 \leqq h \leqq i \leqq k$, and fix h and i. Write

$$(d2)\qquad \sum_{r=1}^{i-1} (\mu^1\ldots\mu^r)\bar{\xi}^r + (\mu^1\ldots\mu^i)\xi = \sum_{r=1}^{h-2} (\mu^1\ldots\mu^r)\bar{\xi}^r + (\mu^1\ldots\mu^{h-1})\tilde{\xi}^{h-1},$$

where

$$(d3)\qquad \tilde{\xi}^{h-1} = \bar{\xi}^{h-1} + \sum_{r=h}^{i-1} (\mu^h\ldots\mu^r)\bar{\xi}^r + (\mu^h\ldots\mu^i)\xi.$$

Then, since by (41) and (51)

$$(d4)\qquad d^r g_{Fh}(\bar{x}, \bar{\xi}^{(r-1)}; \bar{\xi}^r) = 0, \quad (r=0,1,\ldots,h-2),$$

we may write, in view of Lemma 2(a),

$$(d5)\qquad g_{Fh}(\bar{x} + \sum_{r=1}^{i-1}(\mu^1\ldots\mu^r)\bar{\xi}^r + (\mu^1\ldots\mu^i)\xi)$$

$$= d^{h-1}g_{Fh}(\bar{x},\bar{\xi}^{(h-2)};\tilde{\xi}^{h-1})(\mu^1\ldots\mu^{h-1}) + \sum_{r=1}^{h-1} o^r(\mu^r)(\mu^1\ldots\mu^{r-1}).$$

From (d3) it is seen that for any $\mu^r > 0$ $(r=h+1,\ldots,i)$ $\tilde{\xi}^{h-1} \to \bar{\xi}^{h-1}$ as $\mu^h \downarrow 0$. Therefore, Lemma 11 implies that for any $\mu^r > 0$ $(r=h+1,\ldots,i)$ there exists some $\mu^h > 0$ such that

$$(d6)\qquad d^{h-1}g_{Fh}(\bar{x},\bar{\xi}^{(h-2)};\tilde{\xi}^{h-1}) > 0.$$

In view of Appendix B (especially in view of the way that μ^i's are chosen in (b1) and (b2)), we can choose $\mu^i,\mu^{i-1},\ldots,\mu^1$ successively in this order in such a way that these μ's are independent of h and (d1) holds strictly for $h=1,\ldots,i$.

Acknowledgment

I wish to express my thanks to Professors Kenneth J. Arrow and Leonid Hurwicz for their useful discussions during this work.

FOOTNOTES

1/ See [1, theorem 1 and Constraint Qualification w].

2/ See [6, theorems 2 and 3] and Corollaries 2 and 3 below.

3/ See example (29) below. The problem given there is reducible even if the constraint function does not satisfy Constraint Qualification W in [1].

4/ See [5, chapter 6].

5/ In fact, Kuhn and Tucker already suggested that the properties of the directional derivatives of a concave function could be used to examine the possibility of the reduction [4, p. 482].

6/ Eggleston [2, pp. 45-51] and Fenchel [3, pp. 79-88] contain some materials concerning the directional derivatives of a concave function.

7/ See [3, pp. 58, 75, 79-80].

8/ This property will also be obtained if $g(x)$ is defined on an open subset of the n-space.

9/ See [3, pp. 79-80].

10/ See [3, pp. 80-81, 66-67].

11/ The arguments of the function $d^i\varphi(x,\xi^1,\ldots,\xi^{i-1};\xi^i)$ for the case $i = 1$ should read as in $d\varphi(x;\xi^1)$, and the product $(\mu^1\ldots\mu^{i-1})$ for the case $i = 1$ should be regarded as being equal to 1. Such conventions for indexed notations will appear several times in the rest of the paper; they should be interpreted in the most natural

way as the examples above show.

12/ Let $x = (x_1,\ldots,x_n)$. Then, $x \geqq 0$ if $x_i \geqq 0$ $(i=1,\ldots,n)$; $x \geq 0$ if $x \geqq 0$ and $x \neq 0$; and $x > 0$ if $x_i > 0$ $(i=1,\ldots,n)$.

13/ See [3, pp. 5, 40].

REFERENCES

1. Arrow, K. J., L. Hurwicz, and H. Uzawa, "Constraint Qualifications in Maximization Problems," Naval Research Logistics Quarterly, Vol. 8, No. 2, June 1961, pp. 175-191.

2. Eggleston, H. G., Convexity, Cambridge at the University Press, 1963.

3. Fenchel, W., Convex Cones, Sets, and Functions (mimeographed), Princeton University, 1953.

4. Kuhn, H. W. and A. W. Tucker, "Nonlinear Programming," in J. Neyman (Ed.), Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Berkeley: University of California Press, 1950, pp. 481-492.

5. Rockafellar, R. T., Convex Analysis, Princeton University Press, 1970.

6. Uzawa, H., "The Kuhn-Tucker Theorem in Concave Programming," in K. J. Arrow, L. Hurwicz, and H. Uzawa (Eds.), Studies in Linear and Nonlinear Programming, Stanford University Press, 1958, pp. 32-37.

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)
Project on Efficiency of Decision Making in Economic Systems, 1737 Cambridge St., #401, Harvard University, Cambridge, Mass. 02138

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP

3. REPORT TITLE
Regularity Conditions for Concave Programming in Finite Dimensional Spaces

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Report No. 6

5. AUTHOR(S) (First name, middle initial, last name)
Hajime Oniki

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1971	42	6

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
N00014-67-A-0298-0019	Technical Report No. 6
b. PROJECT NO NR 047-004	
c. -----	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) ---
d. -----	

10. DISTRIBUTION STATEMENT
This document has been approved for public release and sale; its publication is unlimited. Reproduction in whole or in part is permitted for any purpose of the United States Government.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY
Logistics & Mathematical Statistics Branch, Dept. of the Navy, Office of Naval Research, Washington, D.C. 20360

13. ABSTRACT

Constrained maximum problems with finitely many variables and finitely many constraints are examined with the assumption that the objective and the constraint functions are concave but not necessarily differentiable. A regularity condition necessary and sufficient for a maximum to be attained and for the problems to be reducible to saddle-point problems is presented. Further, a constraint qualification sufficient for the problems to be regular for any concave objective function is presented, of which Slater-Uzawa's constraint qualifications are special cases.